GENERALIZATIONS OF METRIC SPACES

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INTRODUCTION

The purpose of this study is to examine and relate to one another various topological spaces which are generalized from metric spaces. Much of elementary topology is used with little or no explanation. The concepts of a topological space, open and closed sets, bases for topologies, limit points, closures of sets, continuous functions, and set theory fall into this category.

In Chapter I, we present some of the basic properties of metric and pseudometric spaces, including the notion of the distance between two sets. Although this distance function is not a metric on the entire class of subsets of a pseudometric space, it is a metric if the class of subsets is restricted to those which are closures of one-point sets. The concept of a quotient space is introduced and used to prove this last statement. Chapter I closes with a discussion of the Hausdorff metric for compact subsets of a metric space. Most of the ideas presented in this chapter are discussed in Kelly [6] or Hall and Spencer [4].

Proximity spaces, which are defined and studied in Chapter II, are generalized from metric spaces by dropping the idea of a distance function completely, and postulating certain axioms which are reminiscent of the properties of a metric space. We prove that each proximity space is a completely regular Hausdorff space and that each metric space is a proximity space. Thus the proximity space is a bona fide generalization of a metric space. We introduce the idea of clusters in proximity spaces and use this concept to find a complete topological characterization of a proximity space; namely, a topological space is a proximity space if and only if it is a dense subspace of a compact Hausdorff space. These last theorems are sketched in the paper by S. Leader [7].

Although uniform spaces have been defined in many ways (for example, see Kelly [6] or Gillman and Jerison [2]), we choose to define a uniform space in terms of an arbitrary family of pseudometrics defined on a set. The resulting topology, of course, depends on the family of pseudometrics and so, in general, different families of pseudometrics generate different uniform spaces. We use the idea of uniform continuity to define a uniform structure on a given uniform space; i.e., the uniform structure is the collection of all pseudometrics which are uniformly continuous on the given space. It is shown that the definition of uniform space given herein implies the definitions given by Kelly and Gillman and Jerison. It is proved that a uniform space can be completely characterized topologically as a completely regular space. Hence each proximity space is a Hausdorff uniform space; the converse is also proved. Sufficient conditions for metrizability and pseudometrizability are then obtained. Motivation for this approach is found in Kelly or Gillman and Jerison but the approach in these books is somewhat different from ours.

The thesis closes (Chapter IV) with a brief study of a special type of space referred to in the literature as a generalized metric space. We take a slightly different approach, using a generalized pseudometric. The generalized pseudometric has the properties of a real pseudometric; but the range of the distance function is no longer required to be a subset of the real numbers. Instead, we require merely that the distance function have values in a partially ordered group. For our purpose, we choose a rather specific type. We then prove the result that a topological space is a generalized pseudometric space if and only if it is a uniform space. The basic ideas of generalized metric spaces are presented in Price [8] with one form of our final result to be found in Kalisch [5].

This study is in no way exhaustive. Our treatment of each particular space has been directed to getting the results presented and many results are not even mentioned. Further, there exist other ways of generalizing the concept of distance in a set. For example, see A. Goetz [3] or David Ellis [1].

CHAPTER I

METRIC AND PSEUDOMETRIC SPACES

Let X be an abstract set.

<u>Definition I-1</u>: The function $d : X \times X \to R$ (the real numbers) is called a <u>metric</u> (or distance) <u>function</u> if and only if

(1) $d(x,y) = 0 \Leftrightarrow x = y$.

(2) $d(x,y) = d(y,x)$, for every $x,y \in X$.

(3) $d(x,z) \leq d(x,y) + d(y,z)$, for every $x,y,z \in X$.

The metric d generates a topology for X in the following way: for $x \in X$, neighborhoods of the form

$$V_r(x) = \{y \mid d(x,y) < r, \quad r > 0\}$$

provide a basis for a topology in X. This topological space, denoted by (X,d), is called a metric space.

Note that the metric d has the property that $d(x,y) \geq 0$ for every $x,y \in X$. For replacing y by x in (3) gives

$$d(x,x) \leq d(x,z) + d(z,x)$$

and then (1) and (2) imply that

$$0 \leq 2d(x,z) \Rightarrow d(x,z) \geq 0 \quad \text{for every } x,z \in X.$$

<u>Definition I-2</u>: If the function $\rho : X \times X \to R$ possesses properties (2) and (3) of the preceding definition, but property (1) is replaced by

$$(1') \quad \rho(x,x) = 0 \quad \text{for every } x \in X ,$$

then ρ is called a <u>pseudometric</u>.

That $\rho(x,y) \geq 0$ for every $x,y \in X$ follows as before.

Sets of the form

$$U_r(x) = \{y \mid \rho(x,y) < r, \quad r > 0\}$$

provide a basis for a topology in X. The resulting topological space, denoted by (X,ρ), is called a pseudometric space.

<u>Definition I-3</u>: Let (X,T) be a topological space. If there exists a metric d (or pseudometric ρ) on X such that d (or ρ) generates in X the original topology T, then the space (X,T) is said to be <u>metrizable</u> (or <u>pseudometrizable</u>).

<u>Definition I-4</u>: Two spaces (X,T_1) and (Y,T_2) are <u>homoeomorphic</u> $\Leftrightarrow$ there exists a function $h : X \to Y$ such h is 1-1, onto, continuous and h^{-1} is also continuous. The function h is called a <u>homeomorphism</u>.

<u>Theorem I-1</u>: Let (X,T_1) be a metrizable space and suppose that the space (Y,T_2) is homeomorphic to (X,T_1). Then the space (Y,T_2) is also metrizable.

<u>Proof</u>: Since (X,T_1) and (Y,T_2) are homeomorphic, there exists a homeomorphism $h : X \to Y$.

Let $x, y \in Y$. Then there exist unique points $u, v \in X$ such that $f(u) = x$, $f(v) = y$. Define $d_2(x,y) = d_1(u,v)$, where d_1 is the metric of (X, T_1). Then

(1) $d_2(x,x) = d_1(u,u) = 0$.

(2) $d_2(x,y) = 0 \Rightarrow d_1(u,v) = 0 \Rightarrow u = v \Rightarrow x = y$.

(3) $d_2(x,y) = d_1(u,v) = d_1(v,u) = d_2(y,x)$.

(4) $d_2(x,y) = d_1(u,v) \leq d_1(u,s) + d_1(s,v) = d_2(x,z) + d_2(z,y)$, where $z \in Y$ and $h(s) = z$.

Hence d_2 is a metric on the set Y.

To show that d_2 generates the topology T_2 on Y, let O be an open set in (Y, T_2) with $y \in O$. Then $h^{-1}(O)$ is open in (X, T_1) and there exists an $r > 0$ such that

$$V_r(u) = \{v | d_1(u,v) < r, \ r \in X\} \subset h^{-1}(O),$$

where $u = h^{-1}(y)$.

Thus $h[V_r(u)] \subset O$. Then (by double inclusion),

$$h[V_r(u)] = \{x | d_2(x,y) < r, \ x \in Y\}$$

It follows that O is open in the topology generated by d_2 on Y.

A similar type argument shows that if O is open in the topology generated by d_2 on Y, then O is open in (Y, T_2).

<u>Theorem I-2</u>: Suppose (X,d) is a metric space. Let $(X \times X, T_1)$ be the product space of X with itself and T_1 be the product topology. Then the metric $d : X \times X \to R$ is a continuous function.

Proof: Let O be an open subset of R. Then let $(\overline{x},\overline{y}) \varepsilon d^{-1}(O) \subset X \times X$. Let $h = d(\overline{x},\overline{y}) \varepsilon O$. Choose δ such that $r \varepsilon (h - \delta, h + \delta) \Rightarrow r \varepsilon O$. Let

$$A = \{(x,y) | d(\overline{x},x) < \tfrac{\delta}{3}, \quad d(\overline{y},y) < \tfrac{\delta}{3}\} .$$

Then A is open in the product topology and if $x,y \varepsilon A$, we have

$$d(x,y) \leq d(x,\overline{x}) + d(\overline{x},\overline{y}) + d(y,\overline{y}) < \tfrac{\delta}{3} + h + \tfrac{\delta}{3}$$
$$\Rightarrow d(x,y) < h + \delta .$$

Similarly,

$$d(\overline{x},\overline{y}) \leq d(\overline{x},x) + d(x,y) + d(y,\overline{y}) < d(x,y) + \delta$$
$$\Rightarrow d(x,y) > h - \delta .$$

Hence

$$h - \delta < d(x,y) < h + \delta$$
$$\Rightarrow d(x,y) \varepsilon (h - \delta, h + \delta) \subset O .$$
$$\Rightarrow d(A) \subset O .$$

Hence d is continuous.

Definition I-5: Two subsets A and B of a topological space (X,T) are said to be separated $\Leftrightarrow A \neq \varphi$, $B \neq \varphi$, and $(\overline{A} \cap B) \cup (A \cap \overline{B}) = \varphi$.

Theorem I-3: Two disjoint subsets A and B of a space (X,T) are separated $\Leftrightarrow A \neq \varphi$, $B \neq \varphi$ and each is both open and closed in $A \cup B$.

Proof: If A and B are separated, then $(\overline{A} \cap B) \cup (A \cap \overline{B}) = \varphi \Leftrightarrow \overline{A} \cap B = \varphi$ and $A \cap \overline{B} = \varphi$. Now $\overline{A} \cap B = \varphi \Leftrightarrow B$ contains no limit points of $A \Leftrightarrow B$ is open in $A \cup B \Leftrightarrow A$ is closed in $A \cup B$. Similarly, $A \cap \overline{B} = \varphi \Leftrightarrow A$ is open in $A \cup B \Leftrightarrow B$ is closed in $A \cup B$.

After noting that by definition of separated, $A \neq \varphi$, $B \neq \varphi$, the theorem follows at once.

<u>Definition I-6</u>: A subset A of a topological space (X,T) is <u>connected</u> $\Leftrightarrow A$ is not the union of two separated sets.

<u>Definition I-7</u>: Let (X,d) be a metric space. Suppose A and B are two nonempty subsets of (X,d). The <u>distance between A and B</u>, denoted by $D(A,B)$ is defined as follows:

$$D(A,B) = \inf \{d(x,y) | x \in A,\ y \in B\} .$$

Note that since $d(x,y) \geq 0$ for every $x \in A$, $y \in B$, than $D(A,B)$ always exists and $D(A,B) \geq 0$.

<u>Theorem I-4</u>: Let A and B be nonempty subsets of a metric space (X,d). If $x \in A$, $y \in B$, then

(1) $0 \leq D(A,B) \leq d(x,y)$.

(2) $D(x,A) = 0 \Leftrightarrow x \in \overline{A}$.

(3) Let $H = A \times B$. Define $f : H \to R$ as follows: For any $x \in H$, write $x = (x_1,x_2)$, $x_1 \in A$, $x_2 \in B$; then let $f(x) = d(x_1,x_2)$. Then f is continuous on H.

(4) Further, if $A \neq \varphi$, $B \neq \varphi$, $A \cap B = \varphi$, A and B are compact, then $D(A,B) > 0$. Moreover, $\exists\ a \in A$, $b \in B \ni d(a,b) = D(A,B)$.

<u>Proof</u>: (1) $0 \leq D(A,B)$ follows from the remark before the theorem.

$D(A,B) = \inf \{d(x,y) | x \in A,\ y \in B\} \leq d(x,y)$ for $x \in A$, $y \in B$, follows from the definition of infimum.

(2) $d(x,A) = 0 \iff \inf \{d(x,y) | y \in A\} = 0 . \iff$ every neighborhood of x contains points of A. $\iff x \in \bar{A}$.

(3) Since $A \times B \subset X \times X$, this assertion follows immediately from Theorem I-2.

(4) Suppose $D(A,B) = 0$. Then $\inf \{d(x,y) | x \in A,\ y \in B\} = 0$. Thus, for each $n = 1,2, \ldots,$ there exists $x_n \in A$, $y_n \in B$ for which $d(x_n, y_n) < \frac{1}{n}$. Since A is compact, there exists a subsequence of $\{x_n\}$ which converges to a point of A. Call this limit point x. For notational convenience we will assume that the entire sequence $\{x_n\}$ converges to x. Since B is compact, there exists a subsequence of $\{y_n\}$ which converges to a point y of B. Again, for convenience, we assume that the entire sequence $\{y_n\}$ converges to y. Now since by (3) d is continuous on $A \times B$, we have

$$\lim_{n \to \infty} d(x_n, y_n) = d(x,y) .$$

But $d(x_n, y_n) < \frac{1}{n}$ for $n = 1,2, \ldots,$ gives

$$\lim_{n \to \infty} d(x_n, y_n) = 0 \Rightarrow d(x,y) = 0 \Rightarrow x = y .$$

Since $x \in A$, $y \in B$, then $A \cap B \neq \varphi$, a contradiction, so $D(A,B) > 0$.

Now let $\delta = D(A,B) < \delta + \frac{1}{n}$ for $n = 1,2, \ldots$. Using an identical argument as above, we obtain the existence of $a \in A$, $b \in B$ such that as $n \to \infty$, $x_n \to a$, $y_n \to b$, and $d(x_n, y_n) < \delta + \frac{1}{n}$. Hence

$$\lim_{n\to\infty} d(x_n, y_n) = d(a,b) \leq \delta .$$

But

$$\delta = \inf \{d(x,y) \mid x \in A,\ y \in B\},$$

so

$$\delta \leq d(a,b) \Rightarrow d(a,b) = \delta = D(A,B).$$

Theorem I-5: If A is a fixed subset of a pseudometric space, then $D(A,x)$ is a continuous function of x relative to the pseudometric topology.

Proof: Let $\varepsilon > 0$ be given. Since for $x, z \in X$, $y \in A$,

$$\rho(x,y) \leq \rho(x,z) + \rho(z,y)$$

then,

$$\inf_{y \in A} \{\rho(x,y)\} \leq \rho(x,z) + \inf_{y \in A} \{\rho(z,y)\} .$$

Hence

$$D(x,A) \leq \rho(x,z) + D(z,A)$$

$$\Rightarrow D(x,A) - D(z,A) \leq \rho(x,z) .$$

Interchanging x and z gives

$$D(z,A) - D(x,A) \leq \rho(x,z) .$$

Thus

$$|D(x,A) - D(z,A)| \leq \rho(x,z) .$$

Now if for fixed $x \in X$, we require that $\rho(x,z) < \varepsilon$, we have

$$|D(x,A) - D(z,A)| < \varepsilon$$

$$\Rightarrow D(x,A) \text{ is continuous in } x .$$

<u>Theorem I-6</u>: Let (X,ρ) be a pseudometric space and let $A \subset X$. Then

$$\bar{A} = \{x | D(x,A) = 0\} .$$

<u>Proof</u>: Let $B = \{x | D(x,A) = 0\}$. If we let

$$f(x) = D(x,A),$$

then $f^{-1}(0) = B$. Clearly $A \subset B$ and since f is continuous, B is closed so $\bar{A} \subset B$.

Now suppose $y \in B$. Then $D(y,A) = 0 \Rightarrow$ every neighborhood of y contains points of $A \Rightarrow y \in \bar{A}$. Hence $B \subset \bar{A}$.

It follows then that $\bar{A} = B$.

<u>Definition I-8</u>: A topological space (X,T) is <u>normal</u> $\Leftrightarrow$ for each disjoint pair of closed subsets A and B, there exist disjoint open sets U and V such that $A \subset U$ and $B \subset V$.

<u>Theorem I-7</u>: Each pseudometric space (X,ρ) is normal.

<u>Proof</u>: Let A and B be disjoint closed subsets of a pseudometric space (X,ρ) and let $D(x,A)$ and $D(x,B)$ be the distance from x to A and B respectively.

Let

$$U = \{x | D(x,A) - D(x,B) < 0\}$$

and

$$V = \{x | D(x,A) - D(x,B) > 0\} .$$

That the function f defined by $f(x) = D(x,A) - D(x,B)$ is continuous follows at once from Theorem I-5. Hence U and V are open sets in (X,ρ). Clearly $U \cap V = \varphi$.

Since A is closed, by Theorem I-6, we have

$$A = \{x | D(x,A) = 0\}.$$

Again using Theorem I-6,

$$B = \{x | D(x,B) = 0\}.$$

So if $x \varepsilon A$, then $D(x,B) > 0$. Therefore $D(x,A) - D(x,B) < 0$ if $x \varepsilon A$ and thus $x \varepsilon U$. Hence $A \subset U$. By analogous reasoning, $B \subset V$. Thus (X,ρ) is normal. Several definitions now need to be made.

<u>Definition I-9</u>: If (X,ρ_1) and (Y,ρ_2) are pseudometric spaces, and $f : X \to Y$, then f is an <u>isometry</u> $\Leftrightarrow$ $\rho_1(x,y) = \rho_2(f(x),f(y))$ for every $x,y \varepsilon X$.

Note that if A and B are nonvoid subsets of X such that $D(A,B) > 0$, then we have

$$D(A,B) > D(A,X) + D(X,B) = 0,$$

since the space X is zero distance from each subset of X. Thus the triangular inequality fails; therefore D is not pseudometric.

We will soon see, however, that for a certain subclass of the class of subsets of X, the function D is actually a metric on elements of that subclass. But first, we need some results about quotient spaces.

<u>Definition I-10</u>: An <u>equivalence relation R</u> defined on X is a collection of ordered pairs (x,y), where $x \varepsilon X$, $y \varepsilon X$ which possesses

the following properties: Let x, y, z be arbitrary elements of X. Then

(1) $(x,x) \varepsilon R$.

(2) $(x,y) \varepsilon R \Longleftrightarrow (y,x) \varepsilon R$.

(3) $(x,y) \varepsilon R,\ (y,z) \varepsilon R \Longrightarrow (x,z) \varepsilon R$.

We use the notation xRy interchangeably with $(x,y) \varepsilon R$.

If $x \varepsilon X$, the set

$$R[x] = \{y \mid y \varepsilon X,\ xRy\}$$

is called an equivalence class. Clearly if $z \varepsilon R[x]$, then $x \varepsilon R[z]$ and $R[x] = R[z]$. The collection $\frac{X}{R}$ of all distinct equivalence classes is a disjoint collection of subsets of X whose union is X.

The mapping P defined by the equation

$$P(x) = R[x]$$

is called the projection of X onto the collection $\frac{X}{R}$ of equivalence classes. If $A \subset X$, we denote by $R[A]$ the set

$$R[A] = \{x \mid xRy \text{ for some } y \varepsilon A\} .$$

Let $\mathcal{U} = \{U \mid U \subset \frac{X}{R},\ P^{-1}(U) \text{ is open in } X\}$.

We wish to show that the collection $\mathcal{U}$ is a topology in $\frac{X}{R}$. We note that φ and $\frac{X}{R}$ are both members of $\mathcal{U}$. Let $U = \bigcap_{k=1}^{n} U_k$ be a finite intersection of members of $\mathcal{U}$. Then $P^{-1}[U] = \bigcap_{k=1}^{n} P^{-1}(U_k)$ follows easily by double inclusion. Since $P^{-1}(U_k)$ is open in X, then so is $P^{-1}[U]$. Thus $U \varepsilon \mathcal{U}$.

Now let $U = \bigcup_{i\varepsilon I} U_i$ be an arbitrary union of members of $\mathcal{U}$. Then $P^{-1}[U] = \bigcup_{i\varepsilon I} P^{-1}(U_i)$, again by double inclusion. As before, $P^{-1}(U_i)$ open in $X \Rightarrow P^{-1}[U]$ is open in X also. Therefore $U \varepsilon \mathcal{U}$. Hence $\mathcal{U}$ is a topology for $\frac{X}{R}$.

<u>Definition I-11</u>: If (X,T) is a topological space, R is an equivalence relation on X, P is the projection of X onto the collection $\frac{X}{R}$ of equivalence classes, then the collection $\mathcal{U}$ as defined above is called the <u>quotient topology</u> for $\frac{X}{R}$ (relative to P and T) and the pair $(X,\mathcal{U})$ is called the <u>quotient space</u>.

The quotient topology is easily seen to be the largest topology (i.e., the topology containing the most open sets) for $\frac{X}{R}$ for which P is continuous.

<u>Definition I-12</u>: A function $f : (X,T_1) \to (Y,T_2)$ is called <u>open</u> $\Leftrightarrow$ the image under f of each open set is open. Similarly, f is <u>closed</u> $\Leftrightarrow$ the image under f of each closed set is closed.

<u>Theorem I-8</u>: If the projection P of (X,T) onto the space $(\frac{X}{R}, \mathcal{U}_0)$ is continuous and open, then $\mathcal{U}_0$ is the quotient topology.

<u>Proof</u>: Let U be a subset of $\frac{X}{R}$ such that $P^{-1}(U)$ is open. Then $U = P[P^{-1}(U)]$ is open relative to $\mathcal{U}_0$ since P is an open map. Thus each set open relative to the quotient topology is open relative to $\mathcal{U}_0$. Hence the quotient topology is contained in $\mathcal{U}_0$. Now the quotient topology is the largest topology for X / R

for which P is continuous. Therefore the quotient topology contains $\mathcal{U}_0$. It follows that $\mathcal{U}_0$ is the quotient topology.

<u>Theorem I-9</u>: Let P be the projection of (X,T) onto the quotient space $(X/R, U)$. Then P is an open mapping $\Leftrightarrow$ A is an open subset of X $\Rightarrow$ $R[A]$ is open in (X,T).

<u>Proof</u>: $(\Rightarrow)$ For each open subset A of X, we have

$$R[A] = P^{-1}[P[A]].$$

If P is open, then $P[A]$ is open. Since the mapping $P : (X, T) \to (X/R, U)$ is continuous, then $P^{-1}[P[A]] = R[A]$ is open.

$(\Leftarrow)$ Now suppose for each open set A in (X,T), then $P^{-1}[P[A]]$ is open in (X,T). Then by definition of the quotient topology, $P[A]$ is open. Hence P is an open mapping.

We now return to our earlier discussion of the function $D(A,B)$ defined on subsets of $X \times X$ and show that D is actually a metric on the elements of a certain subclass of the class of subsets of X.

Let (X,ρ) be a pseudometric space and let $\mathcal{D}$ be the family of all sets of the form $\overline{\{x\}}$, i.e., the closure of a one-point set. Since $\overline{\{x\}}$ is the set of all points y for which $\rho(x,y) = 0$, then the family $\mathcal{D}$ of subsets of X is the quotient X/R, where R is the equivalence relation

$$xRy \Leftrightarrow \rho(x,y) = 0 .$$

<u>Theorem I-10</u>: Let (x,ρ) be a pseudometric space and let $\mathcal{D}$ be the family of all subsets of X of the form $\overline{\{x\}}$ for $x \in X$ and for members A and B of $\mathcal{D}$, let $D(A,B) = \inf\{\rho(x,y) \mid x \in A,\ y \in B\}$. Then $(\mathcal{D},D)$ is a metric space whose topology is the quotient topology for $\mathcal{D}$ and the projection of X onto $\mathcal{D}$ is an isometry.

<u>Proof</u>: Suppose $x,y,u,v \in X$ and $u \in \overline{\{x\}}$, $v \in \overline{\{y\}}$. Then $\rho(x,u) = 0 = \rho(y,v)$. Thus

$$\rho(u,v) \leq \rho(u,x) + \rho(x,y) + \rho(y,v) = \rho(x,y) .$$

Further

$$\rho(x,y) \leq \rho(x,u) + \rho(u,v) + \rho(v,y) = \rho(u,v) .$$

Hence

$$\rho(x,y) = \rho(u,v) .$$

Now for

$$A,\ B \in \mathcal{D},$$

$$D(A,B) = \inf\{\rho(x,y) \mid x \in A,\ y \in B\} .$$

But by the above argument, $\rho(x,y)$ is constant for $x \in A$, $y \in B$. Therefore

$$D(A,B) = \rho(x,y), \quad \text{for any } x \in A,\ y \in B .$$

Then, $A, B, C \in \mathcal{D}$, $x \in A$, $y \in B$, $z \in C \Rightarrow$

(1) $D(A,B) = \rho(x,y) = \rho(y,x) = D(B,A)$.

(2) $D(A,C) = \rho(x,z) \leq \rho(x,y) + \rho(y,z) = D(A,B) + D(B,C)$.

(3) If $A = B$, then $D(A,B) = D(A,A) = \rho(x,y) = 0$ for any $x,y \in A$.

(4) If $D(A,B) = 0$, then $\rho(x,y) = D(A,B) = 0$ for every $x \varepsilon A$, $y \varepsilon B$. Hence

$$x \varepsilon \overline{\{y\}} = B \quad \text{for every} \quad x \varepsilon A \Rightarrow A \subset B .$$

Similarly, we conclude that $B \subset A$ so $A = B$.

Therefore D is a metric on $\mathcal{D}$ and clearly the projection $P : (X,\rho) \to (\mathcal{D}, D)$ is an isometry.

We must now show that the topology of $(\mathcal{D},D)$ is the same as the quotient topology for $\mathcal{D}$.

Let U be an open set in (X,ρ). Let x be an arbitrary point in U. Then there exists an $r > 0$ such that

$$\{y | \rho(x,y) < r\} \subset U ,$$

since sets of the indicated form are a basis for the topology of (X,ρ).

Now since $\rho(x,y) = D(A,B)$, where $A = \overline{\{x\}}$, $B = \overline{\{y\}}$, it follows that

$$z \varepsilon \{y | \rho(x,y) < r\} \Leftrightarrow \overline{\{z\}} \varepsilon \left\{\overline{\{y\}} \;\middle|\; D\left(\overline{\{x\}},\overline{\{y\}}\right) < r\right\} .$$

Therefore,

$$\left\{\overline{\{y\}} \;\middle|\; D\left(\overline{\{x\}},\overline{\{y\}}\right) < r\right\} \subset \bigcup_{z \varepsilon U} \overline{\{z\}} .$$

It is easily seen now that the projection P for which $P(x) = \overline{\{x\}}$ is continuous and open from (X,ρ) to $(\mathcal{D},D)$. Then, by Theorem I-8, D generates the quotient topology on $\mathcal{D}$.

The Hausdorff Metric for Subsets

Let (X,d) be a metric space of finite diameter. Let $\mathcal{A}$ be the family of all nonempty closed subsets of (X,d). For $r > 0$ and $A \varepsilon \mathcal{A}$, let

$$V_r(A) = \{x | D(x,A) < r\} .$$

Define, for $A,B \varepsilon \mathcal{A}$,

$$d'(A,B) = \inf \{r | A \subset V_r(B), \quad B \subset V_r(A)\} .$$

Then d' is called the Hausdorff metric. It is not the same as $D(A,B)$ since if $A \cap B \neq \varphi$ and $A \neq B$, then $D(A,B) = 0$. However if $(A, B \varepsilon \mathcal{A})$, we have $d'(A,B) > 0$. Further, the two concepts are different even if $D(A,B) > 0$.

Theorem I-11: $(\mathcal{A},d')$ is a metric space and the function $f : X \to \mathcal{A}$ defined by the equation

$$f(x) = \{x\}$$

is an isometry of X onto a subspace of $\mathcal{A}$.

Proof: We first show that d' is a metric on $\mathcal{A}$. Let $A, B \varepsilon \mathcal{A}$. Suppose $d'(A,B) = 0$ but assume $A \neq B$. Then either there exists an $x \varepsilon A$ for which $x \notin B$ or vice versa. Without loss of generality, assume the former. Then $D(x,B) > 0$. Thus if $A \subset V_r(B)$, then $r \geq D(x,B)$. Then $d'(A,B) = \inf \{r | A \subset V_r(B), \quad B \subset V_r(A)\} \geq D(x,B) > 0$. But $d'(A,B) = 0$ by hypothesis. This contradiction implies $A = B$.

We must now show that $A = B$ implies $d'(A,B) = 0$. By definition, $d'(A,B) = \inf \{r | A \subset V_r(B), \quad B \subset V_r(A)\}$. We note that $B \subset V_r(B)$ for each $r > 0$. Since $A = B$, then $A \subset V_r(B)$ for each $r > 0$. Similarly, $B \subset V_r(A)$ for each $r > 0$. Hence $d'(A,B) = 0$.

It is trivial that $d'(A,B) = d'(B,A)$. Now for the triangular inequality: Let $A, B, C \in \mathcal{A}$. Let

$$R_1 = \{r | A \subset V_r(B), \quad B \subset V_r(A)\} .$$

$$R_2 = \{r | B \subset V_r(C), \quad C \subset V_r(B)\} .$$

$$R_3 = \{r | A \subset V_r(C), \quad C \subset V_r(A)\} .$$

Let $r_1 \in R_1$, $r_2 \in R_2$. We will show that $r_1 + r_2 \in R_3$. Let a be an arbitrary point of A. Then since $r_1 \in R_1$, it follows that $D(A,B) < r_1$. Thus, there exists $b \in B$ for which $d(a,b) < r_1$. Similarly, if $r_2 \in R_2$, then $D(b,C) < r_2$. Again, there exists a $c \in C$ such that $d(b,c) < r_2$. Thus since d is a metric, we have

$$d(a,c) \leq d(a,b) + d(b,c) < r_1 + r_2.$$

Hence

$$\inf \{d(a,x) | x \in C\} \leq d(a,c) < r_1 + r_2 ,$$

or

$$D(a,C) < r_1 + r_2 .$$

Therefore, $a \in V_{r_1 + r_2}(C)$. Since a is an arbitrary point of A,

then $A \subset V_{r_1 + r_2}(C)$.

Similarly, one can show that $C \subset V_{r_1 + r_2}(A)$. So $r_1 + r_2 \varepsilon R_3$. Thus

$$\inf \{r | r \varepsilon R_3\} \leq \inf \{r_1 + r_2 | r_1 \varepsilon R_1, \ r_2 \varepsilon R_2\}$$

and

$$d'(A,C) \leq \inf \{r_1 | r_1 \varepsilon R_1\} + \inf \{ r_2 | r_2 \varepsilon R_2\}$$

or

$$d'(A,C) \leq d'(A,B) + d'(B,C) .$$

Thus d' is a metric and with the topology generated by d', $(\mathcal{A}, d')$ is a metric space.

We now establish the asserted isometry. Let $f(x) = \{x\}$ for $x \varepsilon X$. Since (X,d) is a metric space, all one-point sets are closed. Hence $\{x\} \varepsilon \mathcal{A}$. We will show that if $x,y \varepsilon X$, then

$$d(x,y) = d'\big(\{x\},\{y\}\big) ,$$

so that f is an isometry. Now

$$d'\big(\{x\},\{y\}\big) = \inf \Big\{r | \{x\} \subset V_r\big(\{y\}\big), \{y\} \subset V_r\big(\{x\}\big)\Big\} .$$

Suppose $\{x\} \subset V_{r_1}\big(\{y\}\big)$. Then $D\big(x,\{y\}\big) < r_1$ and further

$$d(x,y) = \inf \Big\{d(x,y) | y \varepsilon \{y\}\Big\} = D\big(x,\{y\}\big) < r_1 .$$

Therefore

$$d(x,y) \leq \inf \Big\{r | \{x\} \subset V_r\big(\{y\}\big)\Big\} ,$$

so

$$d(x,y) \leq \inf \Big\{r | \{x\} \subset V_r\big(\{y\}\big), \{y\} \subset V_r\big(\{x\}\big)\Big\} .$$

Thus

$$d(x,y) \leq d'\big(\{x\},\{y\}\big) .$$

Now let $r_o = d(x,y)$. Then $d(x,y) < r_o + \frac{1}{n}$, for $n = 1,2, \ldots$.

Hence

$$\inf\left\{d(x,y) \mid y \in \{y\}\right\} < r_o + \frac{1}{n}$$

and

$$\inf\left\{d(x,y) \mid x \in \{x\}\right\} < r_o + \frac{1}{n}$$

which gives

$$D\left(x, \{y\}\right) < r_o + \frac{1}{n}$$

$$D\left(y, \{x\}\right) < r_o + \frac{1}{n} ,$$

or

$$x \in V_{r_o + \frac{1}{n}}\left(\{y\}\right), \qquad y \in V_{r_o + \frac{1}{n}}\left(\{x\}\right) ,$$

so that

$$\{x\} \subset V_{r_o + \frac{1}{n}}\left(\{y\}\right), \qquad \{y\} \subset V_{r_o + \frac{1}{n}}\left(\{x\}\right) ,$$

for each $n = 1,2, \ldots$. Therefore

$$d'\left(\{x\},\{y\}\right) < r_o + \frac{1}{n} \quad \text{for } n = 1,2, \ldots .$$

It follows that

$$d'\left(\{x\},\{y\}\right) \leq r_o = d(x,y) .$$

Hence $d(x,y) = d'\left(\{x\},\{y\}\right)$ and **f** is an isometry of X onto the subspace of $\mathcal{A}$ consisting of all one-point sets.

CHAPTER II

PROXIMITY SPACES

A. Basic Properties

Definition II-1: A proximity relation P in a set X is a binary relation $(A,B) \varepsilon P$ (read "A is close to B") defined on subsets of X such that the following axioms hold:

(1) If $(A,B) \varepsilon P$, then $(B,A) \varepsilon P$.

(2) $(A,X) \varepsilon P \Leftrightarrow A \neq \varphi$.

(3) $(A \cup B, C) \varepsilon P \Leftrightarrow$ either $(A,C) \varepsilon P$ or $(B,C) \varepsilon P$.

(4) If for every $E \subset X$, either $(A,E) \varepsilon P$ or $(B,X - E) \varepsilon P$, then $(A,B) \varepsilon P$.

(5) If x and y are points in X and $(x,y) \varepsilon P$, then $x = y$.

The pair (X, P) is called a proximity space. We say A is remote from $B \Leftrightarrow (A,B) \notin P$.

Theorem II-1: If $A \subset C$, $B \subset D$, and $(A,B) \varepsilon P$, then $(C,D) \varepsilon P$.

Proof: By (3), since $(A,B) \varepsilon P$, then $(A \cup C, B) \varepsilon P$ and hence $(A \cup C, B \cup D) \varepsilon P$. But $A \cup C = C$, $B \cup D = D$, so $(C,D) \varepsilon P$.

Theorem II-2: If $A \cap B \neq \varphi$, then $(A,B) \varepsilon P$.

Proof: Suppose $(A,B) \notin P$. Then by (4), there exists $E \subset X$ such that $(A,E) \notin P$ and $(B,X - E) \notin P$. Now $A \cap B \subset A$, so by Theorem II-1, we have $(A \cap B, E) \notin P$. Similarly $A \cap B \subset B \Rightarrow (A \cap B, X - E) \notin P$. Then $(E \cup (X - E), A \cap B) \notin P \Rightarrow (X, A \cap B) \notin P$, by (3). Hence, by (2) $A \cap B = \varphi$, a contradiction.

<u>Theorem II-3</u>: $(\varphi, A) \notin P$ for every $A \subset X$.

<u>Proof</u>: By (2), $(\varphi, X) \notin P$. Now let $A \subset X$; then by Theorem II-1, $(\varphi, A) \notin P$.

<u>Theorem II-4</u>: If there exists $x \varepsilon X$ such that $(x,A) \varepsilon P$ and $(x,B) \varepsilon P$, then $(A,B) \varepsilon P$.

<u>Proof</u>: Suppose $(A,B) \notin P$; then by (4), there exists $E \subset X$ such that $(A,E) \notin P$ and $(B, X - E) \notin P$. Now let $X \varepsilon X$; then either $x \varepsilon X - E$ or $x \varepsilon E$. If $x \varepsilon E$, then $(A,x) \notin P$; If $x \varepsilon X - E$, then $(B,x) \notin P$. Hence, either $(A,x) \notin P$ or $(B,x) \notin P$ for every $x \varepsilon X$, a contradiction.

<u>Definition II-2</u>: If $A \subset X$, the closure of A, denoted by $\overline{A}$, is the set of all points x of X for which $(x,A) \varepsilon P$.

<u>Theorem II-5</u>: If $A \subset X$, then $\overline{A} = \overline{\overline{A}}$.

<u>Proof</u>: Clearly $\overline{A} \subset \overline{\overline{A}}$. Let $x \varepsilon \overline{\overline{A}}$. Then $(x,\overline{A}) \varepsilon P$. We wish to show that $(x,A) \varepsilon P$. Suppose not; then by (4), there exists $E \subset X$ such that $(x,E) \notin P$ and $(A, X - E) \notin P$. If $y \varepsilon \overline{A}$, then $(y,A) \varepsilon P$. Hence $y \notin X - E$ since otherwise, we would have $(A, X - E) \varepsilon P$. Thus $y \varepsilon \overline{A} \Rightarrow y \varepsilon E$. So $\overline{A} \subset E$. Since $(x,E) \notin P$, then $(x,\overline{A}) \notin P$, a contradiction. Therefore $(x,A) \varepsilon P \Rightarrow x \varepsilon \overline{A}$. Hence $\overline{\overline{A}} \subset \overline{A}$.

Theorem II-6: $(A,B) \in P \Leftrightarrow (\bar{A},\bar{B}) \in P$.

Proof: ($\Rightarrow$) This implication follows trivially from Theorem II-1, since $A \subset \bar{A}$, $B \subset \bar{B}$.

($\Leftarrow$) Suppose $(A,B) \notin P$. Then there exists $E \subset X$ for which $(A,E) \notin P$ and $(B, X - E) \notin P$. Arguing as in Theorem II-5, we conclude that $\bar{B} \subset E$. Thus $(A,E) \notin P \Rightarrow (A,\bar{B}) \notin P$. Thus, there exists $E^* \subset X$ such that $(A,E^*) \notin P$ and $(\bar{B}, X - E^*) \notin P$. Again repeating the argument of Theorem II-5, we conclude that $\bar{A} \subset X - E^*$. Therefore, $(\bar{B}, X - E^*) \notin P \Rightarrow (\bar{B},\bar{A}) \notin P$, a contradiction; so $(A,B) \in P$.

Definition II-3: A topological space (X,T) is completely regular $\Leftrightarrow$ for each closed set $C \subset X$ and point $x_o \notin C$, there exists a continuous function $f : X \to [0,1]$ in such a way that $f(x_o) = 0$ and $f(x) = 1$ for each $X \in C$.

Theorem II-7: If we define an open set as the complement of a closed set (a set is closed if it is equal to its closure), the collection of open sets thus defined constitute a topology in X, so that (X,P) is a topological space.

Proof: It is clear that $X = \bar{X}$ and $\varphi = \bar{\varphi}$. Thus both X and φ are open and closed.

We will show that the intersection of any collection of closed sets is closed. Let $C = \bigcap_{i \in I} C_i$ be an intersection of closed sets $\{C_i\}$. Then if $(x,C) \in P$, we have $(X, C_i) \in P$ for each $i \in I \Rightarrow$

$x \varepsilon C_i$ for each $i \varepsilon I$ since C_i is closed. Thus $x \varepsilon C$ so C is closed.

We need to prove now that the union of a finite number of closed sets is closed. Let $C = \bigcup_{i=1}^{n} C_i$ be such a union. Suppose $(x,C) \varepsilon P$. Using (3) a finite number of times gives the existence of at least one C_j for which $(x,C_j) \varepsilon P \Rightarrow X \varepsilon C_j$ since C_j is closed. Thus $x \varepsilon \bigcup_{i=1}^{n} C_i = C$. So C is closed. Hence the collection of open sets thus defined impose a topology on X.

<u>Theorem II-8</u>: Every proximity space (X,P) is a completely regular Hausdorff space.

<u>Proof</u>: We first show that (X,P) is Hausdorff. Let $x,y \varepsilon X$, $x \neq y$. Then $(x,y) \notin P$, by (5). Thus, there exists $E \subset X$ such that $(x,E) \notin P$ and $(y, X - E) \notin P$, by (4). Then by Theorem II-6, $(x,\overline{E}) \notin P$ and $(y, \overline{X - E}) \notin P$. By Theorem II-2, $x \varepsilon X - \overline{E}$ and $y \varepsilon X - \overline{(X - E)}$. Note that $X - \overline{E} \subset X - E$ and $X - \overline{(X - E)} \subset E$. Hence, since $(X - E) \cap E = \varphi$, then $(X - \overline{E}) \cap [X - \overline{(X - E)}] = \varphi$. So (X,P) is Hausdorff.

The following proof that (X,P) is completely regular is constructive and is patterned after the proof of the well-known Urysohn's Lemma.

Let $x_o \varepsilon X$. Let U be an open set (i.e., $X - U$ is closed) such that $x_o \varepsilon U$. Our purpose is to construct a continuous function $f : X \to [0,1]$ such that $f(x_o) = 0$ and $f(y) = 1$ for $y \varepsilon X - U$.

Now $x_o \in U \Rightarrow x_o \notin X - U = \overline{X - U}$. Thus $(x_o, X - U) \notin P$. Therefore there exists $E \subset X$ for which $(X - U, E) \notin P$ and $(x_o, X - E) \notin P$. So $(x_o, \overline{X - E}) \notin P \Rightarrow x_o \in X - \overline{(X - E)}$.

Since $(X - U, E) \notin P$, then $(X - U, \overline{E}) \notin P$. Let $A_{\frac{1}{2}} = X - \overline{(X - E)}$, an open set. Note that $x_o \in A_{\frac{1}{2}}$. Since $(X - U, E) \notin P$ and $A_{\frac{1}{2}} \subset E$, we have $(X - U, A_{\frac{1}{2}}) \notin P \Rightarrow (X - U, \overline{A}_{\frac{1}{2}}) \notin P$. By Theorem II-2, $\overline{A}_{\frac{1}{2}} \cap (X - U) = \varphi$, so $\overline{A}_{\frac{1}{2}} \subset U$. This is the first step in our construction.

Now $x_o \in A_{\frac{1}{2}}$, an open set, so by the procedure just outlined, with $A_{\frac{1}{2}}$ replacing U, we can find an open set $A_{\frac{1}{4}}$ such that $x_o \in A_{\frac{1}{4}}$ and $(X - A_{\frac{1}{2}}, \overline{A}_{\frac{1}{4}}) \notin P$ so that $\overline{A}_{\frac{1}{4}} \subset A_{\frac{1}{2}}$.

Also, since $(X - U, \overline{A}_{\frac{1}{2}}) \notin P$, there exists $E_1 \subset X$ for which $(X - U, E_1) \notin P$ and $(\overline{A}_{\frac{1}{2}}, X - E_1) \notin P$. Define $A_{\frac{3}{4}} = X - \overline{(X - E_1)}$ and then $A_{\frac{3}{4}}$ is open. Further, $X - A_{\frac{3}{4}} = \overline{(X - E_1)}$ gives $(\overline{A}_{\frac{1}{2}}, X - A_{\frac{3}{4}}) \notin P$ $\Rightarrow \overline{A}_{\frac{1}{2}} \subset A_{\frac{3}{4}}$. Noticing that $A_{\frac{3}{4}} \subset E_1$, we have $(X - U, A_{\frac{3}{4}}) \notin P \Rightarrow$ $(X - U, \overline{A}_{\frac{3}{4}}) \notin P \Rightarrow \overline{A}_{\frac{3}{4}} \subset U$.

At the n^{th} stage of the process, we will have 2^{n-1} open sets

$$A_{\frac{1}{2^n}}, A_{\frac{2}{2^n}}, \ldots, A_{\frac{2^n - 1}{2^n}} \quad \text{such that}$$

$$x_o \in A_{\frac{1}{2^n}} \subset A_{\frac{2}{2^n}} \subset \ldots \subset A_{\frac{2^n - 1}{2^n}} \subset U .$$

We continue this process indefinitely so that for each dyadic number $h/2^n$, there exists an open set $A_{h/2^n}$ such that

$$x_o \in A_{\frac{h-1}{2^n}} \subset A_{\frac{h}{2^n}} \subset A_{\frac{h+1}{2^n}} \subset U, \qquad 1 < h < 2^n - 1 .$$

We now define $f\colon X \to [0,1]$ as follows: if $x \in A_t$ for every dyadic fraction t, then $f(x) = 0$; otherwise $f(x) = \sup\{t \mid x \notin A_t,$ t a dyadic fraction$\}$. Clearly $x_o \in A_t$ for every dyadic fraction t. Hence $f(x_o) = 0$.

Further, since $A_t \subset U$ for every dyadic fraction t, then if $X \in U$, $x \notin A_t$ for any dyadic fraction t. So for $x \in U$,

$$f(x) = \sup\{t \mid x \notin A_t, \ t \text{ a dyadic fraction}\} = 1.$$

Note the trivial fact that $0 \leq f(x) \leq 1$ for every $x \in X$. So $f : X \to [0,1]$.

We must now establish continuity of f. Let $x \in X$.

<u>Case I</u>: $(0 < f(x) < 1)$ Let $\varepsilon > 0$ be given. Choose two dyadic fractions t_1 and t_2 so that

$$f(x) - \varepsilon < t_1 < f(x) < t_2 < f(x) + \varepsilon .$$

Then consider the open set $A_{t_2} - \overline{A}_{t_1}$. Note that $x \in A_{t_2}$ and since there exists a dyadic fraction t^* such that $t_1 < t^* < f(x)$, we have $x \notin A_{t^*}$. Since $\overline{A}_{t_1} \subset A_{t^*}$, we have $x \notin \overline{A}_{t_1}$. Thus $x \in A_{t_2} - \overline{A}_{t_1}$.

Let y be any point of $A_{t_2} - \overline{A}_{t_1}$. Then

$$f(y) = \sup\{t \mid t \text{ is a dyadic fraction}, \ x \notin A_t\} < t_2$$

and

$$f(y) \geq t_1 .$$

Then

$$f(x) - \varepsilon < t_1 < f(y) < t_2 < f(x) + \varepsilon$$

$$\Rightarrow -\varepsilon < f(y) - f(x) < \varepsilon$$

$$\Rightarrow |f(y) - f(x)| < \varepsilon .$$

So f is continuous at x.

Case II: $f(x) = 0$ or $f(x) = 1$ is handled similarly.

So (X,P) is completely regular.

The next theorem shows that the metric spaces are actually a subclass of the proximity spaces.

Theorem II-9: Let (X,d) be a metric space. Then define $(A,B) \varepsilon P$ $\Leftrightarrow D(A,B) = 0$. Then (X,P) is a proximity space.

Proof: (1) $(A,B) \varepsilon P \Rightarrow (B,A) \varepsilon P$ follows from $D(A,B) = D(B,A)$.

(2) $(A,X) \varepsilon P \Leftrightarrow D(A,X) = 0 \Leftrightarrow A \neq \phi$.

(3) $(A \cup B, C) \varepsilon P \Leftrightarrow D(A \cup B, C) = 0 \Leftrightarrow$ either $D(A,C)=0$ or $D(B,C) = 0 \Leftrightarrow (A,C) \varepsilon P$ or $(B,C) \varepsilon P$.

(4) Suppose $(A,B) \notin P$; then $D(A,B) = h > 0$. Let

$$E = \{x | D(x,A) < \frac{h}{2}\} .$$

Then $A \subset E$ and $B \subset X - E$. Note that $D(B, E) \geq \frac{h}{2}$ and $D(A, X - E) \geq \frac{h}{2}$. Hence $(B,E) \notin P$ and $(A,X - E) \notin P$. So (4) holds.

(5) If $(x,y) \varepsilon P$; then $D(\{x\},\{y\}) = d(x,y) = 0$. Since d is a metric, $x = y$.

Since closures of sets in (X,d) and (X,P) are equivalent, it is easily seen that the topologies in (X,d) and (X,P) are equivalent.

Note that in step (5), our proof would have failed if d had been only a pseudometric. It is easily seen that if the pseudometric space (X,ρ) is to be a proximity space, it is necessary and sufficient that ρ be a metric.

Definition II-4: A topological space (X,T) is normal $\Leftrightarrow$ for any two disjoint closed sets, C_1 and C_2, there exist disjoint open sets, O_1 and O_2, in (X,T) such that $C_1 \subset O_1$ and $C_2 \subset O_2$.

Theorem II-10: Each compact Hausdorff space (X,T) is normal.

Proof: This is a standard theorem in topology. The reader may consult Kelly [6], page 141, Theorem 5-9, for the proof.

Theorem II-11: Each compact Hausdorff space (X,T) is a proximity space if we define

$$(A,B) \in P \Leftrightarrow \bar{A} \cap \bar{B} \neq \varphi .$$

Proof: (1) $(A,B) \in P \Rightarrow \bar{A} \cap \bar{B} \neq \varphi \Rightarrow \bar{B} \cap \bar{A} \neq \varphi \Rightarrow (B,A) \in P$.

(2) $(A,X) \in P \Leftrightarrow \bar{A} \cap X \neq \varphi \Leftrightarrow \bar{A} \neq \varphi \Leftrightarrow A \neq \varphi$.

(3) $(A \cup B, C) \in P \Leftrightarrow (\overline{A \cup B}) \cap \bar{C} \neq \varphi \Leftrightarrow (\bar{A} \cup \bar{B}) \cap \bar{C} \neq \varphi$ $\Leftrightarrow (\bar{A} \cap \bar{C}) \cup (\bar{B} \cap \bar{C}) \neq \varphi \Leftrightarrow$ either $\bar{A} \cap \bar{C} \neq \varphi$ or $\bar{B} \cap \bar{C} \neq \varphi \Leftrightarrow$ $(A,C) \in P$ or $(B,C) \in P$.

(4) Suppose $(A,B) \notin P$. Then $\bar{A} \cap \bar{B} = \varphi$. Now since (X,T) is compact and Hausdorff, then (X,T) is normal. So there exist open sets

E and F, for which

$$\bar{A} \subset E, \quad \bar{B} \subset F, \quad \text{and} \quad E \cap F = \varphi .$$

Then $\bar{A} \cap (X - E) = \varphi \Rightarrow (A, X - E) \notin P$ (since $X - E$ is closed). Further $\bar{B} \cap (X - F) = \varphi \Rightarrow \bar{B} \cap \bar{E} = \varphi$ since $\bar{E} \subset X - F$. Hence $(B,E) \notin P$. Thus (4) holds.

(5) Suppose $(x,y) \in P$. Then since X is Hausdorff, $(x,y) \in P \Rightarrow \overline{\{x\}} \cap \overline{\{y\}} \neq \varphi \Rightarrow x \cap y \neq \varphi \Rightarrow x = y$.

B. Clusters in Proximity Spaces

We will now consider special subclasses (called clusters) of the class of all subsets of a space X. The use of these special subclasses will enable us to prove a very important result about proximity spaces; namely, a complete topological characterization of a proximity space.

Definition II-5: A cluster Ω from a proximity space (X,P) is a class of subsets of X satisfying:

(a) If $A,B \in \Omega$, then $(A,B) \in P$.

(b) If $(A,C) \in P$ for every $C \in \Omega$, then $A \in \Omega$.

(c) If $A \cup B \in \Omega$, then either $A \in \Omega$ or $B \in \Omega$.

Intuitively, a cluster might be thought of as a maximal class of subsets of X which are "pairwise close."

We now state some elementary results about clusters. In the following discussion, we shall sometimes refer to a one-point set $\{x\}$ as a point x.

<u>Theorem II-12</u>: Let (X,P) be a proximity space and let $x \varepsilon X$. Let $\Omega = \{A | A \subset X, (A,x) \varepsilon P\}$. Then Ω is a cluster.

<u>Proof</u>: That Ω satisfies (a) follows immediately from Theorem II-4.

Suppose that $(A,B) \varepsilon P$ for every $A \varepsilon \Omega$. The point $x \varepsilon \Omega$ since $(x,x) \varepsilon P$ by Theorem II-2. Thus $(x,B) \varepsilon P \Longleftrightarrow B \varepsilon \Omega$, so (b) is satisfied.

Now suppose $A \cup B \varepsilon \Omega$. Then by axiom (3), we have immediately that either $A \varepsilon \Omega$ or $B \varepsilon \Omega$ so (c) is satisfied.

<u>Theorem II-13</u>: Suppose a point x is a member of a cluster Ω; then

$$\Omega = \{A | A \subset X, (A,x) \varepsilon P\} .$$

<u>Proof</u>: Suppose $B \varepsilon \Omega$. Then by (a), $(B,x) \varepsilon P$. Thus $B \varepsilon \{A | A \subset X, (A,x) \varepsilon P\} \Rightarrow \Omega \subset \{A | A \subset X, (A,x) \varepsilon P\}$.

Conversely, suppose $B \varepsilon \{A | A \subset X, (A,x) \varepsilon P\}$. Then $(x,B) \varepsilon P$. Now, by (a), if $A \varepsilon \Omega$, then $(A,x) \varepsilon P$, and it follows that $(A,B) \varepsilon P$ for every $A \varepsilon \Omega$. Now by (b), $B \varepsilon \Omega$. Thus

$$\{A | A \subset X, (A,x) \varepsilon P\} \subset \Omega$$

and equality follows.

Now suppose the points x and y are both members of a cluster Ω. Then by (a), $(x,y) \varepsilon P$ and thus $x = y$. Hence, there exists at most one point x so that x is a member of Ω.

We shall soon see that there exist clusters which do not have a one-point set as a member. Indeed, the noncompact proximity spaces have at least one cluster Ω which fails to have a one-point set as a member of Ω.

That X is a member of every cluster from X follows from (b). So by (c), for any $E \subset X$, either E or $X - E$ is a member of any given cluster.

The theorems to follow are more profound, and the proofs correspondingly more involved. In fact, we find it necessary to apply some form of the axiom of choice. The form we choose to use is Zorn's Lemma. For a discussion of partially ordered sets and Zorn's Lemma, the reader is referred to Taylor [9], pp. 39-40.

Theorem II-14: If $(A_o,B) \varepsilon P$, then there exists a cluster Ω for which $A_o \varepsilon \Omega$ and $B \varepsilon \Omega$.

Proof: Let A_o and B be subsets of X for which $(A_o,B) \varepsilon P$. Let $\mathcal{A}$ be a collection of subsets of X satisfying the following properties:

(1) $A_o \varepsilon \mathcal{A}$.

(2) If $\{A_1, A_2, \ldots, A_n\}$ is any finite subcollection of $\mathcal{A}$, then $(B, \bigcap_{i=1}^{n} A_i) \varepsilon P$.

We denote, with the letter $\mathcal{a}_o$, the class of all such collections $\mathcal{A}$ satisfying (1) and (2).

Note that $\mathcal{a}_o$ is nonempty since by hypothesis $\{A_o\} \varepsilon \mathcal{a}_o$. We seek to apply Zorn's Lemma to the collection $\mathcal{a}_o$.

The elements of $\mathcal{a}_o$ are partially ordered by the relation set inclusion. Further, if $\mathcal{a}_1$ is a subclass of $\mathcal{a}_o$, i.e., $\mathcal{a}_1 \subset \mathcal{a}_o$, and $\mathcal{A}_1$ is linearly ordered, then clearly $\bigcup\{\mathcal{A} \mid \text{all } \mathcal{A} \varepsilon \mathcal{A}_1\} \supset \mathcal{A}$ for any $\mathcal{A} \varepsilon \mathcal{A}_1$.

Now let $\{A_1, A_2, \ldots, A_n\}$ be any finite subcollection of $\bigcup\{\mathcal{A} \mid$ all $\mathcal{A} \varepsilon \mathcal{A}_1\}$. Then there exists $\{\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n\}$, so that $A_k \varepsilon \mathcal{A}_k \subset \mathcal{A}_1$. Since $\mathcal{A}_1$ is linearly ordered, there exists a number $\bar{N}$, where $1 \leq \bar{N} \leq n$, for which $\mathcal{A}_{\bar{N}} \supset \mathcal{A}_j$ for $j = 1,2, \ldots, n$; $j \neq \bar{N}$. Thus $A_j \varepsilon \mathcal{A}_{\bar{N}}$, $j = 1,2, \ldots, n$. Now $\mathcal{A}_{\bar{N}} \varepsilon \mathcal{A}_1 \subset \mathcal{A}_0$, so $(B, \bigcap_{i=1}^{n} A_i) \varepsilon P$. Thus $\bigcup\{\mathcal{A} \mid$ all $\mathcal{A} \varepsilon \mathcal{A}_1\} \varepsilon \mathcal{A}_0$ follows easily. Zorn's Lemma now applies and guarantees the existence of at least one maximal element $\mathcal{A}_0 \varepsilon \mathcal{A}_0$.

Now let

$$\Omega = \{C \mid C \subset X, \ (C,A) \varepsilon P \text{ for each } A \varepsilon \mathcal{A}_0\} .$$

Notice that $A_0 \varepsilon \mathcal{A}_0$. Let A be any element of $\mathcal{A}_0$. Then $(A_0 \cap A, B) \varepsilon P$ since $\mathcal{A}_0$ possesses property (2). So $A_0 \cap A \neq \varphi$ by Theorem II-3. By Theorem II-2, $(A_0, A) \varepsilon P$. Hence $A_0 \varepsilon \Omega$.

Moreover, since $(A,B) \varepsilon P$ for each $A \varepsilon \mathcal{A}_0$, we have that $B \varepsilon \Omega$. It remains only to show that Ω is indeed a cluster, i.e., we must show that Ω satisfies axioms (a), (b), and (c) of Definition II-5.

(a) Now, let $G \varepsilon \Omega$, $H \varepsilon \Omega$ and suppose that $(G,H) \notin P$. Then by (4), there exists $E \subset X$ such that $(G,E) \notin P$ and $(H, X - E) \notin P$. Since for every $A \varepsilon \mathcal{A}_0$, we have $A \cap E \subset E$ and $A - E \subset X - E$, then $(G, A \cap E) \notin P$ and $(H, A - E) \notin P$. So $A \cap E \notin \mathcal{A}_0$ and $A - E \notin \mathcal{A}_0$.

We assert that there exist subsets A_1 and A_2 in $\mathcal{A}_0$ for which $(A_1 \cap E, B) \notin P$ and $(A_2 - E, B) \notin P$. For suppose not; then for every

$A \in \mathcal{A}_o$, we have $(A \cap E, B) \in P$. Let $\{A_1, A_2, \ldots, A_n\}$ be any finite subcollection of $\mathcal{A}_o$. Then

$$\bigcap_{k=1}^{n} (A_k \cap E) = (\bigcap_{k=1}^{n} A_k) \cap E .$$

Now $\bigcap_{k=1}^{n} A_k \in \mathcal{A}_o$ since $\mathcal{A}_o$ is maximal. Then by our supposition, $(\bigcap_{k=1}^{n} (A_k \cap E), B) \in P$. It follows easily that $A \cap E \in \mathcal{A}_o$ for every $A \in \mathcal{A}_o$ or else $\mathcal{A}_o$ is not maximal. But this contradicts our conclusion in the previous paragraph that $A \cap E \notin \mathcal{A}_o$ for each $A \in \mathcal{A}_o$. Hence there exists A_1 for which $A_1 \in \mathcal{A}_o$ and $(A_1 \cap E, B) \notin P$. By noting that $A - E = A \cap (X - E)$, we see that an analogous argument will show that there exists $A_2 \in \mathcal{A}_o$ with $(A_2 - E, B) \notin P$.

Now let $A = A_1 \cap A_2$. Since $\mathcal{A}_o$ is maximal, then $A \in \mathcal{A}_o$. Then $A \subset A_1 \Rightarrow A \cap E \subset A_1 \cap E$ and also $A - E \subset A_2 - E$. Therefore $(A \cap E, B) \notin P$ and $(A - E, B) \notin P$. Thus, by (3), $(A, B) \notin P$. But this contradicts $A \in \mathcal{A}_o$. So our assumption that $(G, H) \notin P$ is false and Ω satisfies (a).

(b) Let A_1, A_2 be any two elements of $\mathcal{A}_o$. Then $(A_1 \cap A_2, B) \in P$. Thus $A_1 \cap A_2 \neq \varphi$ so $(A_1, A_2) \in P$. So each set in $\mathcal{A}_o$ is close to every other set of $\mathcal{A}_o$, and therefore $\mathcal{A}_o \subset \Omega$. Now if $(A, C) \in P$ for every $C \in \Omega$, then it follows that $(A, C) \in P$ for every $C \in \mathcal{A}_o$. Thus $A \in \Omega$ and Ω satisfies (b).

(c) Suppose now that $G \cup H \in \Omega$, but $G \notin \Omega$. Then $(G \cup H, A) \in P$ for every $A \in \mathcal{A}_o$ and

$(G,A_1) \notin P$ for some $A_1 \notin \mathcal{A}_0$. Now $A_1 \cap A \in \mathcal{A}_0$ for every $A \in \mathcal{A}_0$, because $\mathcal{A}_0$ is maximal. Therefore $(G \cup H, A_1 \cap A) \in P$ for every $A \in \mathcal{A}_0$. From $A_1 \cap A \subset A_1$, it follows that $(A_1 \cap A, G) \notin P$ for every $A \in \mathcal{A}_0$. By (3), we now conclude that $(A_1 \cap A, H) \in P$. Hence $(A,H) \in P$ for every $A \in \mathcal{A}_0$. Thus $H \in \Omega$ and Ω obeys (c).

This completes the proof.

<u>Definition II-6</u>: A family $\mathcal{A}$ of subsets of X is said to have the finite intersection property $\Leftrightarrow$ the intersection of the members of any finite subfamily of $\mathcal{A}$ is nonempty.

<u>Theorem II-15</u>: A topological space (X, T) is compact $\Leftrightarrow$ each family of closed sets which possess the finite intersection property has a nonempty intersection.

<u>Proof</u>: This is a basic theorem in topology. For the proof, the reader may consult Kelly [6], Theorem 5-1, page 136.

<u>Theorem II-16</u>: A proximity space (X,P) is compact if and only if every cluster from X possesses a one-point set.

<u>Proof</u>: Suppose (X,P) is compact, and let Ω be a cluster from X. Suppose Ω does not contain a one-point set. Then by (b), for any $x \in X$, there exists a set $C \in \Omega$ for which $(x,C) \notin P$. By (4), there exists a set $E_x \subset X$ so that $(E_x,C) \notin P$ and $(x, X - E_x) \notin P$. Then clearly $x \notin X - E_x$ so $x \in E_x$. Now $(x, X - E_x) \notin P \Rightarrow (x, \overline{X - E_x}) \notin P \Rightarrow x \in X - \overline{(X - E_x)}$. Denote by O_x the open set $X - \overline{(X - E_x)}$.

Clearly $O_x \subset E_x$ and therefore $(O_x, C) \notin P$. Further, it follows that $O_x \notin \Omega$.

Now for each $x \in X$, such an O_x exists and $x \in O_x$. Thus, $\bigcup_{x \in X} O_x$ contains X. Since X is compact, a finite subcollection of the class $\{O_x | x \in X\}$ covers X. Since $O_x \notin \Omega$ for any $x \in X$, we may use (c) and mathematical induction to see that any finite union of members of $\{O_x | x \in X\}$ does not belong to Ω. It follows that X is not a member of Ω, which gives a contradiction.

Hence there exists some point $x \in X$ for which $\{x\}$ is a member of Ω.

Now suppose that for any cluster Ω from X, there exists $x \in X$ for which $\{x\} \in \Omega$. Let ψ be a class of closed subsets of X such that the intersection of any finite number of members of ψ is nonvoid. To show that X is compact, we will show that the intersection of all sets of ψ is nonvoid.

Using Zorn's Lemma as in Theorem II-14, there exists a class of closed subsets ψ_o such that $\psi \subset \psi_o$ and ψ_o is maximal with respect to the finite intersection property. Define the class Ω to consist of all sets whose closures belong to ψ_o. Now $E \in \Omega \Leftrightarrow \overline{E} \in \psi_o$ and since ψ_o is maximal, $\overline{E} \in \psi_o \Leftrightarrow \overline{E} \cap F \neq \varphi$ for every $F \in \psi_o$. So

(*) $\quad E \in \Omega \Leftrightarrow \overline{E} \cap F \neq \varphi$ for every $F \in \psi_o$.

Now by Theorem II-14, if $(A,B) \in P$, there exists a cluster Ω_1 to which both A and B belong. Now by hypothesis, there exists $x \in X$ for which $\{x\} \in \Omega_1$. Thus $(x,A) \in P$ and $(x,B) \in P \Rightarrow x \in \overline{A}$, $x \in \overline{B}$

$\Rightarrow \bar{A} \cap \bar{B} \neq \varphi$. Now $\bar{A} \cap \bar{B} \neq \varphi \Rightarrow (A,B) \in P$ by Theorem II-4. Hence $(A,B) \in P \Leftrightarrow \bar{A} \cap \bar{B} \neq \varphi$. Combining this result with (*) gives

(**) $E \in \Omega \Leftrightarrow (E,F) \in P$ for every $F \in \psi_0$.

We propose to show that Ω is a cluster.

(a) Let $C_1, C_2 \in \Omega$. Then $\bar{C}_1, \bar{C}_2 \in \psi_0$. Hence $\bar{C}_1 \cap \bar{C}_2 \neq \varphi$. Thus $(C_1, C_2) \in P$.

(b) Clearly $\psi_0 \subset \Omega$. Now, if $(A,C) \in P$ for every $C \in \Omega$, it follows that $(A,C) \in P$ for every $C \in \psi_0$. Using (**), we have $A \in \Omega$.

(c) Suppose now that $E \cup F \in \Omega$ with $E \notin \Omega$. We must prove that $F \in \Omega$. We have $\overline{E \cup F} = \bar{E} \cup \bar{F} \in \psi_0$, but $\bar{E} \notin \psi_0$. Now if $\bar{E} \notin \psi_0$, then $\bar{E} \cap B_1 = \varphi$ for some $B_1 \in \psi_0$, because ψ_0 is maximal with respect to the finite intersection property. Now for every $A \in \psi_0$, $B_1 \cap A \neq \varphi$ and $B_1 \cap A$ is closed. The maximality of ψ_0 implies that $B_1 \cap A \in \psi_0$. So

$$(\bar{E} \cup \bar{F}) \cap (B_1 \cap A) \neq \varphi \quad \text{for every } A \in \psi_0.$$

Now

$$\bar{E} \cap B_1 = \varphi \Rightarrow \bar{E} \cap (B_1 \cap A) = \varphi.$$

Then

$$(\bar{E} \cup \bar{F}) \cap (B_1 \cap A) \neq \varphi \Rightarrow \bar{F} \cap (B_1 \cap A) \neq \varphi \quad \text{for every } A \in \psi_0.$$

Thus

$$\bar{F} \cap A \neq \varphi \quad \text{for every } A \in \psi_0$$

and hence

$$(F,A) \in P \text{ for every } A \in \psi_0 \Rightarrow F \in \Omega \text{ by (**)}.$$

Hence Ω is a cluster.

Now by hypothesis, there exist $x \varepsilon X$ for which $\{x\} \varepsilon \Omega$. By (a), $(x,C) \varepsilon P$ for every $C \varepsilon \Omega$. Since $\psi_o \subset \Omega$, then $(x,A) \varepsilon P$ for every $A \varepsilon \psi_o$. Each set in ψ_o is closed; therefore $x \varepsilon A$ for every $A \varepsilon \psi_o$. Hence the intersection of all sets of ψ_o is nonvoid. Thus (X,P) is compact.

Corollary: In a compact proximity space (X,P), $(A,B) \varepsilon P \Leftrightarrow \bar{A} \cap \bar{B} \neq \varphi$.

Proof: $(A,B) \varepsilon P \Rightarrow$ there exists a cluster Ω to which both A and B belong. Since (X,P) is compact, there exists $x \varepsilon X$ so that $\{x\} \varepsilon \Omega$. So $(x,A) \varepsilon P$ and $(x,B) \varepsilon P \Rightarrow x \varepsilon \bar{A}$, $x \varepsilon \bar{B} \Rightarrow \bar{A} \cap \bar{B} \neq \varphi$.

Conversely, $\bar{A} \cap \bar{B} \neq \varphi \Rightarrow (\bar{A},\bar{B}) \varepsilon P \Rightarrow (A,B) \varepsilon P$, by previous theorems.

Theorem II-17: Let X be a subspace of a proximity space Y. Then every cluster ψ from X is part of a unique cluster Ω from Y, where Ω is the class of all subsets of Y which are close to every set in ψ.

Proof: Let ψ be a cluster from X. Let Ω be as indicated in the theorem. Since ψ is a cluster, then by (a), each set in ψ is close to every other set in ψ. Thus $\psi \subset \Omega$.

We shall show that Ω is a cluster from Y.

(1) Let $(A,B \not\varepsilon P$. Then there exists a set $E \subset X$ so that $(A,E) \not\varepsilon P$ and $(B, X - E) \not\varepsilon P$. Now let C be any set in ψ. Noting that

$$C = (C \cap E) \cup (C - E),$$

then, since ψ is a cluster, we have by (c), that either $C \cap E$ or $C - E$ belongs to ψ. If $C \cap E \in \psi$, then $(A, E) \notin P \Rightarrow (A, C \cap E) \notin P$. Hence $A \notin \Omega$. If $C - E \in \psi$, then $(B, X - E) \notin P \Rightarrow (B, C - E) \notin P$. Thus $B \notin \Omega$. So either A or B does not belong to Ω.

(2) If A is close to every $C \in \Omega$, then $(A,B) \in P$ for every $B \in \psi$ since $\psi \subset \Omega$. Hence $A \in \Omega$.

(3) Suppose $A \cup B \in \Omega$, but $A \notin \Omega$. Then there exists $D \in \psi$ so that $(A,D) \notin P$. By (4), there exists a set $E \subset Y$ for which $(E,A) \notin P$ and $(Y - E, D) \notin P$. Now let C be any set in ψ. Then since $C - E \subset Y - E$, $(C - E, D) \notin P$. Since $D \in \psi$, then $C - E \notin \psi$. Moreover, since $(C - E) \cup (C \cap E) = C \in \psi$, then $C \cap E \in \psi$. We are assuming $A \cup B \in \Omega$; thus $(A \cup B, C \cap E) \in P$.

However, $(A,E) \notin P \Rightarrow (A, C \cap E) \notin P$. So by (3), $(B, C \cap E) \in P$ and therefore $(B,C) \in P$. But C was an arbitrary set in ψ. Hence $B \in \Omega$.

Therefore, Ω is a cluster. We next show that Ω is unique.

Suppose Ω_1 is a cluster from Y which contains ψ and for which $\Omega_1 \neq \Omega$. Then if $D \in \Omega_1$, it follows that D is close to every set in $\psi \subset \Omega_1$. Hence $D \in \Omega \Rightarrow \Omega_1 \subset \Omega$.

Now let $C \in \Omega$. Since Ω is a cluster, C is close to every set in Ω and since $\Omega_1 \subset \Omega$, C is close to every set in Ω_1. Then by (b), $C \in \Omega_1 \Rightarrow \Omega \subset \Omega_1$.

Therefore $\Omega = \Omega_1$ and Ω is unique.

<u>Theorem II-18</u>: If X is a subspace of a proximity space Y, every cluster Ω from Y to which X belongs contains a unique cluster ψ from X, where ψ consists of all subsets of X which belong to Ω.

<u>Proof</u>: The proof follows the pattern of Theorem II-17, so we sketch the proof.

Let Ω be a cluster from Y for which $X \varepsilon \Omega$. Let ψ be defined as in the theorem.

(a) If $A,B \varepsilon \psi$, then since $\psi \subset \Omega$; $A,B \varepsilon \Omega$. Thus $(A,B) \varepsilon P \Rightarrow \psi$ satisfies (a).

(b) Suppose $A \subset X$, $A \not\varepsilon \psi$. Then $A \varepsilon \Omega$. Since Ω is a cluster, there exists $C \varepsilon \Omega$ for which $(A,C) \not\varepsilon P$. Hence, there exists $E \subset Y$ for which $(A,E) \varepsilon P$ and $(C, Y - E) \not\varepsilon P$. So $Y - E \not\varepsilon \Omega$. Further, $(C, X - E) \not\varepsilon P$ so $X - E \not\varepsilon \Omega$. Now $X \varepsilon \Omega$ so, by (c), $X \cap E \varepsilon \Omega$. Since $(A,E) \not\varepsilon P$, we have $(A, X \cap E) \not\varepsilon P$. Now $(X \cap E) \varepsilon \Omega$ so (b) is satisfied by ψ.

(c) Suppose that $A \cup B \varepsilon \psi$ with $A \not\varepsilon \psi$. Then $A \cup B \varepsilon \Omega$ and $A \not\varepsilon \Omega$. Hence $B \varepsilon \Omega$ because Ω is a cluster. Therefore $B \varepsilon \psi$ so ψ satisfies (c).

Thus ψ is a cluster from X.

For uniqueness, suppose ψ_1 is a cluster from X and $\psi_1 \subset \Omega$. If $D \varepsilon \psi_1$, then $D \subset X$ and $D \varepsilon \Omega$. Hence $D \varepsilon \psi \Rightarrow \psi_1 \subset \psi$.

If $B \varepsilon \psi$, then $(B,C) \varepsilon P$ for every $C \varepsilon \psi$. Hence $(B,D) \varepsilon P$ for every $D \varepsilon \psi_1$ since $\psi_1 \subset \psi$. So $B \varepsilon \psi_1 \Rightarrow \psi \subset \psi_1$.

Hence $\psi = \psi_1$ and ψ is unique.

C. A Topological Characterization of a Proximity Space

Let $A \subset X$, where (X,P) is a proximity space. For any $x \varepsilon A$, the collection of all sets close to x form a cluster. We denote by A' the family of all clusters determined by points of A. Thus a cluster Ω belongs to A' if and only if Ω is the collection of all subsets of X close to some point x of A.

Let A'' be the collection of all clusters of which A is a number. Note that since X is a member of every cluster from X, then X'' is the collection of all clusters from X.

Theorem II-19: If A' and A'' are defined as above, then $A' \subset A''$.

Proof: Let $\Omega \varepsilon A'$. Then there exists an $x \varepsilon A$ such that Ω is the family of all sets close to x. Since $x \varepsilon A$, we have $(x,A) \varepsilon P$. Thus $A \varepsilon \Omega$. By definition of A'', we have $\Omega \varepsilon A''$. Hence $A' \subset A''$.

Suppose that $\Omega \varepsilon X'$. We have already observed that the point $x \varepsilon X$ which determines Ω is unique. Thus the mapping $F : X \to X'$ which carries a point x onto the collection of all sets close to x is a one-to-one mapping. Note that F is also onto.

Definition II-7: Two proximity spaces (X_1,P_1) and (X_2,P_2) are isomorphic if and only if there exists a one-to-one function $T : X_1 \to X_2$, which is onto, and for any two subsets A and B of X_1 satisfies the condition

$$(A,B) \varepsilon P_1 \text{ if and only if } (T(A), T(B)) \varepsilon P_2.$$

We say that the functions T and T^{-1} <u>preserve proximity</u>.

We note that isomorphism as defined above is stronger than homeomorphism between proximity spaces. For if T preserves proximity, we have: if $x \in \bar{A} \subset X_1$, then $(X,A) \in P_1$ if and only if $(T(x), T(A))$ $\in P_2$. Thus $T(x) \in \overline{T(A)}$. Similarly if $y \in \bar{C} \subset X_2$, then $T^{-1}(y) \in$ $\overline{T^{-1}(C)}$. So both T and T^{-1} are continuous. It follows that whenever (X_1,P_1) and (X_2,P_2) are isomorphic, then they are also homeomorphic.

The converse is true if (X_1,P_1) and (X_2,P_2) are compact proximity spaces. Suppose (X_1,P_1) and (X_2,P_2) are homeomorphic and each is compact. If $(A,B) \in P_1$, then there exists $x \in X_1$ for which $(x,A) \in P_1$, $(x,B) \in P_1$. Thus $x \in \bar{A} \cap \bar{B}$. Let h be the homeomorphism mapping X_1 onto X_2. Then $h(x) \in \overline{h(A)} \cap \overline{h(B)}$ by continuity of h. It follows that $(h(x), h(A)) \in P_2$, $(h(x), h(B)) \in P_2$. So h preserves proximity. By similar argument, h^{-1} also preserves proximity.

The following example shows that without compactness the results of the previous paragraph need not hold.

<u>Example</u>: Let $X_1 = (-\infty,0) \cup (0,\infty) \subset E_1$ and $X_2 = (-\infty,1) \cup (1,\infty)$ $\subset E_1$. Define $h : X_1 \to X_2$ by

$$h(x) = \begin{cases} x + 1, & \text{if } x > 0 \\ x - 1, & \text{if } x < 0 \end{cases}.$$

Then h is a homeomorphism. Let P be the proximity relation induced by the metric $d(x,y) = |x - y|$. Then (X_1,P) and (X_2,P) are proximity spaces.

Let

$$A = (-1,0), \quad B = (0,1).$$

Then

$$h(A) = (-2,-1), \quad h(B) = (1,2) .$$

Clearly $(A,B) \varepsilon P$ but $(h(A), h(B)) \notin P$. Hence h fails to preserve proximity. So (X_1,P) and (X_2,P) are homeomorphic, but not isomorphic. Note that neither (X_1,P) nor (X_2,P) are compact proximity spaces. Note also that the cluster of which both A and B are members does not contain a one-point set.

Definition II-8: Let $A \subset X$ and let $\Gamma \subset X''$. Then we say that A absorbs Γ if and only if A belongs to every cluster in Γ, that is, if and only if $\Gamma \subset A''$.

Note that Γ above is just a collection of clusters from X.

Now we intend to define a proximity relation P^* on subsets of X'' so that (X'',P^*) will be a proximity space. Let Γ_1,Γ_2 be subsets of X''.

Then we agree that $(\Gamma_1,\Gamma_2) \varepsilon P^*$ if and only if $(A,B) \varepsilon P$ whenever A absorbs Γ_1, B absorbs Γ_2. That is, $(\Gamma_1,\Gamma_2) \varepsilon P^*$ if and only if $\Gamma_1 \subset A''$, $\Gamma_2 \subset B''$ implies $(A,B) \varepsilon P$.

Theorem II-20: The relation P^* satisfies the properties of a proximity relation.

Proof: (1) Suppose $(\Gamma_1,\Gamma_2) \varepsilon P^*$; then if $\Gamma_1 \subset A''$, $\Gamma_2 \subset B''$, we have $(A,B) \varepsilon P$. Therefore $(B,A) \varepsilon P$, so $(\Gamma_2,\Gamma_1) \varepsilon P^*$.

(2) We note that $(\Gamma_1, X'') \varepsilon P^* \Leftrightarrow \Gamma_1 \subset A''$, $X'' \subset B''$ $\Rightarrow$ $(A,B) \varepsilon P$. Now $X'' \subset B''$ $\Leftrightarrow$ B belongs to every cluster from X. It follows that $X'' \subset B''$ $\Leftrightarrow$ B is close to every nonvoid subset of X. If $\Gamma_1 \neq \varphi$, then $\Gamma_1 \subset A''$ $\Rightarrow$ $A \neq \varphi$. Thus $(A,B) \varepsilon P$ and $(\Gamma_1, X'') \varepsilon P^*$.

If Γ_1 is void, then $\Gamma_1 \subset \varphi''$. But $(\varphi, X) \notin P$ so $(\Gamma_1, X'') \notin P^*$.

(3) Suppose $(\Gamma_1 \cup \Gamma_2, \Gamma_3) \varepsilon P^*$ and suppose $(\Gamma_1, \Gamma_3) \notin P^*$. If $\Gamma_2 \subset B''$ and $\Gamma_3 \subset C''$, we shall show that $(B,C) \varepsilon P$. Since $(\Gamma_1, \Gamma_3) \notin P^*$, then there exists sets $A, D \subset X$ for which $\Gamma_1 \subset A''$, $\Gamma_3 \subset D''$ and $(A,D) \notin P$. Then there exists a set $E \subset X$ so that $(A,E) \notin P$ and $(D, X - E) \notin P$. Therefore $(D, C - E) \notin P$. Since $\Gamma_3 \subset D''$, $C - E$ belongs to no cluster in Γ_3. But $\Gamma_3 \subset C''$ so C belongs to every cluster in Γ_3. Now $(C - E) \cup (C \cap E) = C$ so by (c), $C \cap E$ belongs to every cluster in Γ_3. Hence $\Gamma_3 \subset (C \cap E)''$. Also by (c), we have $(A \cup B)'' = A'' \cup B''$. Thus, if $\Gamma_1 \subset A''$, $\Gamma_2 \subset B''$, then $\Gamma_1 \cup \Gamma_2 \subset (A \cup B)''$. Hence $(\Gamma_1 \cup \Gamma_2, \Gamma_3) \varepsilon P^* \Rightarrow (A \cup B, C \cap E) \varepsilon P$. Now $(A,E) \notin P$, so $(A, C \cap E) \notin P$. Hence $(B, C \cap E) \varepsilon P$ and so $(B,C) \varepsilon P$. Therefore $\Gamma_2 \subset B''$, $\Gamma_3 \subset C''$ $\Rightarrow$ $(B,C) \varepsilon P$. By definition of P^*, we have $(\Gamma_1, \Gamma_3) \varepsilon P^*$.

Now suppose conversely that $\Gamma_1, \Gamma_2, \Gamma_3$ are subsets of X'', with $(\Gamma_1, \Gamma_2) \varepsilon P^*$. Suppose $\Gamma_1 \cup \Gamma_3 \subset D''$ and $\Gamma_2 \subset C''$. Then $\Gamma_1 \subset D''$ and since $(\Gamma_1, \Gamma_2) \varepsilon P^*$, we have $(C,D) \varepsilon P$. Hence $(\Gamma_1 \cup \Gamma_3, \Gamma_2) \varepsilon P^*$.

So P^* satisfies axiom (3).

(4) Suppose $(\Gamma_1, \Gamma_2) \notin P^*$. Then there exist subsets A and B of X for which $(A,B) \notin P$ and for which $\Gamma_1 \subset A''$, $\Gamma_2 \subset B''$. Since P

satisfies (4), there exists an $E \subset X$ such that $(A,E) \notin P$ and $(B, X - E) \notin P$. Since $\Gamma_2 \subset B''$ and $(X - E, B) \notin P$, then $X - E$ belongs to no cluster in Γ_2. But $X = (X - E) \cup E$ is a member of every cluster in Γ_2. So by (c), E is a member of every cluster in Γ_2; hence $\Gamma_2 \subset E''$. Since $\Gamma_1 \subset A''$ and $(A,E) \notin P$, then $(\Gamma_1, E'') \notin P^*$. By definition of E'', E belongs to no cluster in $X'' - E''$. Now $X = E \cup (X - E)$ belongs to every cluster in X'', so by (c), $(X - E)$ belongs to every cluster in $X'' - E''$. Therefore $X'' - E'' \subset (X - E)''$. Since $\Gamma_2 \subset B''$ and $(X - E, B) \notin P$, then $(\Gamma_2, X'' - E'') \notin P^*$. So P^* satisfies axiom (4).

(5) Let Ω_1, Ω_2 be clusters from X. Suppose that $(\Omega_1, \Omega_2) \in P^*$. We will show that $\Omega_1 = \Omega_2$. For every $A,B \subset X$ such that $\Omega_1 \in A''$, $\Omega_2 \in B''$, we have $(A,B) \in P$. Now $\Omega_1 \in A''$, $\Omega_2 \in B'' \Leftrightarrow A \in \Omega_1$, $B \in \Omega_2$. So $(A,B) \in P$ for every $B \in \Omega_2$ and for every $A \in \Omega_1$. Hence $A \in \Omega_2$ and $B \in \Omega_1$. Thus $\Omega_1 = \Omega_2$.

Conversely, suppose $\Omega_1 = \Omega_2$. Then if $\Omega_1 \in A''$, $\Omega_2 \in B''$, we have $A \in \Omega_1$, $B \in \Omega_2$. Thus $(A,B) \in P$ and hence $(\Omega_1, \Omega_2) \in P^*$. So P^* satisfies axiom (5).

We have shown that P^* is indeed a proximity relation. Thus (X'', P^*) is a proximity space and by Theorem II-8 is a completely regular Hausdorff space. We will now prove that (X,P) is isomorphic to (X', P^*) as a proximity space. Note here that (X', P^*) is a subspace of (X'', P^*) since $X' \subset X''$ and the same proximity relation is used.

Theorem II-21: Every proximity space (X,P) is isomorphic to dense subspace of a compact Hausdorff space (Y,P'). Further, (Y,P') is uniquely determined by (X,P) up to an isomorphism.

Proof: We will show that X' is dense in X''. Suppose that A,B are subsets of X. Then $B' \subset A'' \Leftrightarrow \Omega \varepsilon B' \Rightarrow \Omega \varepsilon A''$. Now $\Omega \varepsilon B' \Leftrightarrow$ there exists a $c \varepsilon B$ such that Ω is the class of all subsets of X which are close to c. Now $\Omega \varepsilon A'' \Leftrightarrow A \varepsilon \Omega$. But $A \varepsilon \Omega \Leftrightarrow (A,c) \varepsilon P$. So $\Omega \varepsilon A'' \Leftrightarrow c \varepsilon \bar{A}$. So $B' \subset A'' \Leftrightarrow B \subset \bar{A}$. Now if ψ is a cluster from X, we have $(B',\psi) \varepsilon P^* \Leftrightarrow (B,D) \varepsilon P$ whenever $D'' \supset \psi$. Hence $(B',\psi) \varepsilon P^* \Leftrightarrow B \varepsilon \psi$, by (b). But $(B',\psi) \varepsilon P^* \Leftrightarrow \psi \varepsilon \bar{B}'$ and $B \varepsilon \psi \Leftrightarrow \psi \varepsilon B''$. So $\psi \varepsilon \bar{B}' \Leftrightarrow \psi \varepsilon B''$. Hence $B'' = \bar{B}'$. Taking $B = X$, We have $X'' = \bar{X}'$. So X' is dense in X''.

We now show that X is isomorphic to X'. Note that $(A',B') \varepsilon P^* \Leftrightarrow A' \subset C'$, $B' \subset D' \Rightarrow (C,D) \varepsilon P$. But $A' \subset C''$, $B' \subset D'' \Leftrightarrow A \subset \bar{C}$ and $B \subset \bar{D}$, by our argument in the previous paragraph. So $(A', B') \varepsilon P^* \Leftrightarrow A \subset \bar{C}$ and $B \subset \bar{D} \Rightarrow (C,D) \varepsilon P$. Now $A \subset \bar{C}$, $B \subset \bar{D} \Rightarrow (C,D) \varepsilon P \Leftrightarrow (A,B) \varepsilon P$. So $(A',B') \varepsilon P^* \Leftrightarrow (A,B) \varepsilon P$. Thus the function $T : X \to X'$ preserves proximity in both directions. As noted earlier, T is one-to-one and onto. So X is isomorphic to X'.

We next show that (X'',P^*) is a compact space. To do this, we use Theorem II-16, and show that for each cluster Ω'' from X'', there exists $\Omega \varepsilon X''$ for which $\Omega \varepsilon \Omega''$. (Note that Ω'' is <u>not</u> a class of subsets of X, but a class of subsets of X'' satisfying axioms (a), (b), and (c).) Let Ω'' be any cluster from X''. Then since X' is dense in X'', we have that X' is close to every subset of X'' and hence X' belongs to every cluster from X. So by Theorem II-18, there exists a unique cluster Ω' from X' so that $\Omega' \subset \Omega''$. Now if $A' \varepsilon \Omega'$ and $B' \varepsilon \Omega'$, we have $(A',B') \varepsilon P^*$, or equivalently, $(A,B) \varepsilon P$.

Consider the collection Ω of all sets $A \subset X$ such that $A' \varepsilon \Omega'$. We assert that Ω is a cluster from X. For if $A,B \varepsilon \Omega$, then $A' \varepsilon \Omega'$, $B' \varepsilon \Omega'$ so $(A',B') \varepsilon P^*$ and therefore $(A,B) \varepsilon P$. So (a) is satisfied.

Suppose now that $(A,C) \varepsilon P$ for every $C \varepsilon \Omega$. Then $(A',C') \varepsilon P^*$ for every $C' \varepsilon \Omega'$ and thus $A' \varepsilon \Omega'$ and $A \varepsilon \Omega$. So (b) is satisfied.

Now let $A \cup B \varepsilon \Omega$ with $A \notin \Omega$. Then $(A \cup B)' \varepsilon \Omega'$ and $A' \notin \Omega'$. Since $(A \cup B)' = A' \cup B'$, then $A' \cup B' \varepsilon \Omega'$. By (c), since Ω' is a cluster and $A' \notin \Omega'$, we have $B' \varepsilon \Omega'$ and thus $B \varepsilon \Omega$. So Ω satisfies (c). Hence Ω is a cluster from X with the property that $A \varepsilon \Omega \Leftrightarrow A' \varepsilon \Omega'$.

Since (by Theorem II-18), for any cluster Ω'' from X'', there exists a unique cluster Ω' from X' such that $\Omega' \subset \Omega''$, we now have a unique cluster Ω from X such that $\Omega' = \{A' | A \varepsilon \Omega\}$. Now $B \varepsilon \Omega \Leftrightarrow \Omega \varepsilon B''$. Since $\overline{B}' = B''$, we have $B \varepsilon \Omega \Leftrightarrow (\Omega, B') \varepsilon P^*$. But $B \varepsilon \Omega \Leftrightarrow B' \varepsilon \Omega'$. Hence $B' \varepsilon \Omega' \Leftrightarrow (B',\Omega) \varepsilon P^*$. Now $\Omega \varepsilon X''$ and Ω is close to every member of Ω'. So by Theorem II-17, since $X' \subset X''$ and $\Omega' \subset \Omega''$, then $\Omega \varepsilon \Omega''$. Thus, given a cluster Ω'' from X'', there exists $\Omega \varepsilon X''$ for which $\Omega \varepsilon \Omega''$. Hence X'' is compact, by Theorem II-16.

Finally, we show that X'' is unique up to an isomorphism. Suppose Y is any compact space such that $\overline{X} = Y$. Then by Theorem II-16, if Ω is a cluster from Y, then there exists an $x \in Y$ such that $\{x\} \in \Omega$. Now $\overline{X} = Y \Rightarrow X$ belongs to every cluster from Y. So by Theorem II-18, there exists a unique cluster ψ from X such that $\psi \subset \Omega$. Further, if ψ is a cluster from X, then by Theorem II-17, there exists a unique cluster Ω from Y for which $\psi \subset \Omega$ and moreover, since Y is compact, there exists $x \in Y$ such that $\{x\} \in \Omega$. So each point in Y corresponds to a unique cluster from X and each cluster from X corresponds to a unique point in Y. Hence we have a one-to-one (onto) correspondence between points of Y and members of X''.

We contend that the correspondence thus defined is an isomorphism. For let $A,B \subset Y$ and $(A,B) \in P$. Then from $\overline{A \cap X} = \overline{A} \cap \overline{X} = \overline{A} \cap Y = \overline{A}$ and $\overline{B \cap X} = \overline{B}$, we have $(A,B) \in P \Leftrightarrow (A \cap X, B \cap X) \in P$. But $(A \cap X, B \cap X) \in P \Leftrightarrow ((A \cap X)', (B \cap X)') \in P^*$ and since $(A \cap X)'' \supset (A \cap X)'$ and $(B \cap X)'' \supset (B \cap X)'$, we have $((A \cap X)'', (B \cap X)'') \in P^*$.

Conversely, let Γ_1, Γ_2 be subsets of X'' so that $(\Gamma_1, \Gamma_2) \in P^*$. Now $(\Gamma_1, \Gamma_2) \in P^* \Leftrightarrow \Gamma_1 \subset A''$ and $\Gamma_2 \subset B'' \Rightarrow (A,B) \in P$, where $A,B \subset Y$. Since $\overline{A \cap X} \supset A$, then $A \cap X$ is a member of each cluster in X'' corresponding to points of A. Hence $(A \cap X)'' \supset \Gamma_1$. Similarly $(B \cap X)'' \supset \Gamma_2$. Then $((A \cap X)'', (B \cap X)'') \in P^*$. But since $\overline{(A \cap X)'} = (A \cap X)''$ and $\overline{(B \cap X)'} = (B \cap X)''$, we have $((A \cap X)', (B \cap X)') \in P^*$, or equivalently, $(A \cap X, B \cap X) \in P$. Thus $(A,B) \in P$. So Y is isomorphic to X''.

Theorem II-22: A topological space (X,T) admits a proximity relation if and only if (X,T) is a dense subspace of a compact Hausdorff space.

Proof: The theorem follows easily from Theorem II-11 and Theorem II-21.

CHAPTER III

UNIFORM SPACES

We come now to a second major generalization of a metric space. We use the word basis in the following sense.

Definition III-1: A collection of subsets $\{B_\alpha | \alpha \varepsilon J\}$ of a set X is a basis for a topology in X provided

(1) $\bigcup \{B_\alpha | \alpha \varepsilon J\} = X$ and

(2) if $x \varepsilon B_1 \cap B_2$, where B_1, $B_2 \varepsilon \{B_\alpha\}$, then there exists an element B_3 of $\{B_\alpha\}$ such that $x \varepsilon B_3 \subset B_1 \cap B_2$.

Let $\{S_\alpha | \alpha \varepsilon J\}$ be a collection of subsets of X. If the collection of all finite intersections of members of $\{S_\alpha\}$ form a basis, then $\{S_\alpha\}$ is called a subbasis.

A topology T is defined in X by means of a basis $\{B_\alpha\}$ as follows: a subset O of X is a member of T if and only if O is a union of members of $\{B_\alpha\}$. It can easily be shown that T thus defined satisfies the properties required of a topology in X.

Now, let W be a family of psudometrics on a set X. We consider spherical neighborhoods of the form

$$(*) \quad V(x,\rho,r) = \{y | \rho(x,y) < r,\ x \varepsilon X,\ \rho \varepsilon W,\ r > 0\} .$$

Definition III-2: If $W = \{\rho_\alpha\}$ is a family of pseudometrics on X, then the collection of sets of the form $(*)$ is a subbasis for a topology in X. This topology $\mathcal{U}$ is called the uniform topology in X

generated by W. The pair $(X,\mathcal{U})$ is called a uniform space. If $\mathcal{U}$ is generated by $\{\rho_\alpha\}$ we also use $\left(X,\{\rho_\alpha\}\right)$ interchangeably with $(X,\mathcal{U})$.

To justify our definition, we must show that the collection $\{B_\alpha\}$ of all finite intersections of sets of the form (*) is indeed a basis.

It is evident that the collection $\{B_\alpha\}$ covers X. Let B_α, $B_\beta \in \{B_\alpha\}$. Consider $B = B_\alpha \cap B_\beta$. Noting that B is just a finite intersection of sets of the form (*), it follows that $B \in \{B_\alpha\}$. Hence the collection $\{B_\alpha\}$ satisfies Definition III-1.

Note that if (X,d) is a metric space, then it is easily seen that (X, d) is the uniform space generated by the collection consisting of the single metric d. The set of real numbers with the usual metric will be useful as a uniform space.

We now define the concept of uniform continuity.

Definition III-3: Let $\left(X, \{\rho_\alpha\}\right)$ and $\left(Y, \{\rho'_\beta\}\right)$ be two uniform spaces. Then the function $f : X \to Y$ is called uniformly continuous provided that if $\rho'_\beta \in \{\rho'_\beta\}$ and $\varepsilon > 0$ are arbitrarily chosen, there exists a finite subcollection $\{\rho_k\}$, $k = 1,2,\ldots, n$, of $\{\rho_\alpha\}$ and a $\delta > 0$ for which

$$\rho'_\beta\,(f(x_1),\ f(x_2)) < \varepsilon \quad \text{whenever} \quad \max_{1\leq k\leq n} \{\rho_k(x_1,x_2)\} < \delta\ .$$

Let us now consider the product topology in the product space $X \times X$ where $\left(X, \{\rho_\alpha\}\right)$ is a uniform space. We will show that the product topology is a uniform topology which is inherited from the uniform topology in $\left(X, \{\rho_\alpha\}\right)$ in a quite natural way.

For each ρ_α $\{\rho_\alpha\}$, define

$$R_\alpha(x,y) = \rho_\alpha(x_1,y_1) + \rho_\alpha(x_2,y_2), \quad \text{where } x = (x_1,x_2),$$

$y = (y_1,y_2)$, and x_i, y_i ε X, $i = 1, 2$.

It is a trivial consequence of the definition that R_α is a pseudometric on X x X. That the uniform topology generated by the family $\{R_\alpha\}$ on X x X is identical with the product topology is also trivial. For if $V_1(x_1, \rho_\alpha, r_1)$ and $V_2(x_2, \rho_\beta, r_2)$ are spherical neighborhoods in $(X,\{\rho_\alpha\})$, then $V_1 \times V_2$ is a subbasis element of the product topology. But clearly $U_1(x,R_\alpha,r_1) \cap U_2(x,R_\beta,r_2)$ is a subset of $V_1 \times V_2$. For if $y \, \varepsilon \, U_1 \cap U_2$, then whenever $R_\alpha(x,y) < r_1$, we have $\rho_\alpha(x_1,y_1) < r_1$ and whenever $R_\beta(x,y) < r_2$, it follows that $\rho_\beta(x_2,y_2) < r_2$. Thus if $y_1 \varepsilon V_1$, $y_2 \, \varepsilon \, V_2$, then $y \, \varepsilon \, V_1 \times V_2$.

Conversely, let $U(x,R_\alpha,r)$ be a spherical neighborhood in $(X \times X, \{R_\alpha\})$. Then it is easily seen that $V_1(x_1,\rho_\alpha, \frac{r}{2}) \times V_2(x_2,\rho_\alpha, \frac{r}{2})$ is a subset of U.

We sum up in the following theorem.

<u>Theorem III-1</u>: Let $(X,\{\rho_\alpha\})$ be a uniform space. Then the product topology in X x X is a uniform topology. Thus, the product space is a uniform space.

<u>Theorem III-2</u>: Let $(X, \{\rho_\alpha\})$ be a uniform space. Then if $\rho \, \varepsilon \, \{\rho_\alpha\}$, ρ is uniform continuous on X x X relative to the product (uniform) topology:

<u>Proof</u>: Let $x = (x_1,x_2)$, $y = (y_1,y_2)$ be two points in the product space X x X. Let $\varepsilon > 0$ be given. We shall show there exists a generating pseudometric R_α of the product space and $\delta > 0$ such that if

$R_\alpha(x,y) < \delta$, then $|\rho(x_1,x_2) - \rho(y_1,y_2)| < \varepsilon$.

Let $R(x,y) = \rho(x_1,y_1) + \rho(x_2,y_2)$. We have already seen that R is one of the pseudometrics which generate the product space. Now let $\delta = \varepsilon$; then suppose $R(x,y) < \varepsilon$ for $x,y \in X \times X$

Then by the triangular inequality

$$\rho(x_1,x_2) - \rho(y_1,y_2) \leq \rho(x_1,y_1) + \rho(y_2,x_2) = R(x,y) < \varepsilon$$

Similarly

$$\rho(y_1,y_2) - \rho(x_1,x_2) \leq R(x,y) < \varepsilon$$

Therefore

$$|\rho(x_1,x_2) - \rho(y_1,y_2)| < \varepsilon .$$

Definition III-4: Let $(X, \{\rho_\alpha\})$ be a uniform space whose topology is generated by the collection of pseudometrics $\{\rho_\alpha\}$. Let S be the set of all pseudometrics which are uniformly continuous on $X \times X$. Then S is called the uniform structure for $(X,\{\rho_\alpha\})$.

We shall need the following lemma.

Lemma: If ρ_1 and ρ_2 are pseudometrics on a set X, then $\rho = \max\{\rho_1,\rho_2\}$ is a pseudometric on X also.

Proof: The symmetry, non-negativity, and vanishing properties of pseudometrics follow immediately from those of ρ_1 and ρ_2.

Since $\rho_k(x,z) \leq \rho_k(x,y) + \rho_k(y,z) \leq \rho(x,y) + \rho(y,z)$ for $k = 1,2$, then

$$\rho(x,z) \leq \rho(x,y) + \rho(y,z).$$

and the triangular inequality holds.

Theorem III-3: Let $(X, \{\rho_\alpha\})$ be a uniform space and let S be the uniform structure generated by $\{\rho_\alpha\}$. Then

(1) if $\rho_1, \rho_2 \in S$, then $\rho = \max \{\rho_1, \rho_2\} \in S$ and

(2) if ρ_1 is a pseudometric and if for every $\varepsilon > 0$, there exists a $\rho_2 \in S$ and $\delta > 0$ such that

$$\rho_1(x,y) \leq \varepsilon \quad \text{whenever} \quad \rho_2(x,y) \leq \delta$$

for every $x,y \in X$, then $\rho_1 \in S$.

Proof: (1) Let $\rho_k \in S$, $k = 1,2$. Let $\rho = \max \{\rho_1, \rho_2\}$. Let $\varepsilon > 0$ be given and let $x = (x_1, x_2)$, $y = (y_1, y_2)$. By the proof of Theorem III-2, there exists an $R_k \in \{R_\alpha\}$ and $\delta_k > 0$ such that if $R_k(x,y) < \delta_k$, then $|\rho_k(x_1,x_2) - \rho_k(y_1,y_2)| < \varepsilon$ for $k = 1,2$.

Let $\delta = \min \{\delta_1, \delta_2\}$. Then with $R = \max \{R_1, R_2\}$, we have

if $R(x,y) < \delta$, then $R_k(x,y) < \delta$ and hence

$$|\rho_k(x_1,x_2) - \rho_k(y_1,y_2)| < \varepsilon \quad \text{for } k = 1,2.$$

Now if

$$\rho(x_1,x_2) - \rho(y_1,y_2) = \rho_k(x_1,x_2) - \rho_k(y_1,y_2) ,$$

for either $k = 1$ or $k = 2$, then clearly

$$|\rho(x_1,x_2) - \rho(y_1,y_2)| < \varepsilon .$$

If $\rho(x_1,x_2) - \rho(y_1,y_2) = \rho_k(x_1,x_2) - \rho_j(y_1,y_2)$, $k \neq j$, then either (a) $\rho_k(x_1,x_2) \geq \rho_j(y_1,y_2)$ or (b) $\rho_k(x_1,x_2) < \rho_j(y_1,y_2)$. Suppose, for definiteness, that (a) holds. The case where (b) holds is handled similarly. In case (a), we have

$$\rho(y_1,y_2) = \rho_j(y_1,y_2) \geq \rho_k(y_1,y_2) .$$

Thus,

$$0 \leq \rho_k(x_1,x_2) - \rho_j(y_1,y_2) \leq \rho_k(x_1,x_2) - \rho_k(y_1,y_2) ,$$

from which it follows

$$|\rho(x_1,x_2) - \rho(y_1,y_2)| \leq \rho_k(x_1,x_2) - \rho_k(y_1,y_2)| < \varepsilon .$$

Therefore if

$$R(x,y) < \delta$$

then

$$|\rho (x_1,x_2) - \rho(y_1,y_2)| < \varepsilon$$

so ρ is uniformly continuous on $X \times X$.

(2) Now let ρ_1 be any pseudometric on $X \times X$. Let $\varepsilon > 0$ be given. Assume there exists $\rho_2 \varepsilon S$ and $\delta > 0$ such that

$$\rho_1(x_1,x_2) \leq \varepsilon/2$$

whenever

$$\rho_2(x_1,x_2) \leq \delta$$

for every $x_1,x_2 \varepsilon X$. Now

$$R_2(x,y) = \rho_2(x_1,x_2) + \rho_2(y_1,y_2)$$

where $x = (x_1,x_2)$, $y = (y_1,y_2)$. Now suppose $R_2(x,y) \leq \delta$. Then

$$\rho_2(x_1,x_2) \leq \delta \quad \text{and} \quad \rho_2(y_1,y_2) \leq \delta.$$

Hence

$$|\rho_1(x_1,x_2) - \rho_1(y_1,y_2)| \leq \rho_1(x_1,x_2) + \rho_1(y_1,y_2) \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Therefore ρ_1 is uniformly continuous on $X \times X$. Hence $\rho_1 \varepsilon S$.

This last theorem shows that our definition of a uniform structure implies the definition of uniform structure given in Gillman and Jerison [2].

Theorem III-4: If $\{\rho_\alpha\}$ and $\{\rho'_\beta\}$ generate the same uniform structure on X, then $(X,\{\rho_\alpha\})$ and $(X,\{\rho'_\beta\})$ are the same topologically.

Proof: Let S be the uniform structure generated by $\{\rho_\alpha\}$. We will show that (X, S) and $(X,\{\rho_\alpha\})$ are topologically equivalent. The result of the theorem will then be obvious.

Clearly, every open set in $(X, \{\rho_\alpha\})$ is open in (X, S) since $\{\rho\ \} \subset S$.

Let $\rho \varepsilon S$. Then a subbasis element of (X, S) is a set of the form

$$V(x_1, \rho, r) = \{x_2 | \rho(x_1,x_2) < r, \rho \varepsilon S, r > 0\} .$$

If $\rho \varepsilon \{\rho_\alpha\}$, then obviously $V(x_1, \rho, r)$ is a subbasis element of $(X, \{\rho_\alpha\})$ and is hence open in $(X, \{\rho_\alpha\})$. Even if $\rho \not\varepsilon \{\rho_\alpha\}$, ρ is still uniformly continuous on $X \times X$ (relative to the product topology generated by $\{\rho_\alpha\}$). Let $x = (x_1,x_2)$, $y = (y_1,y_2)$ and $R_\alpha(x, y) = \rho_\alpha(x_1,y_1) + \rho_\alpha(x_2,y_2)$. Then there exists $\{R_k\}_{k=1}^n \subset \{R_\alpha\}$ and $\delta > 0$ such that if

$$R = \max \{R_1,R_2, \dots, R_n\},$$

we have $|\rho(x_1,x_2) - \rho(y_1,y_2)| < r$ whenever $R(x,y) < \delta$.

In particular, we choose $x_2 = y_1 = y_2$, and then

$$|\rho(x_1,x_2) - 0| = \rho(x_1,x_2) < r$$

if

$$R_k(x,y) = \rho_k(x_1,y_1) + \rho_k(x_2,y_2) = \rho_k(x_1,x_2) < \delta$$

for each $k = 1,2,\ldots, n$.

So if $\rho_k(x_1,x_2) < \delta$ for each $k = 1,2,\ldots, n$, then $\rho(x_1,x_2) < r$. Thus

$$\bigcap_{k=1}^{n} V(x_1,\rho_k, \delta) \subset V(x_1, \rho, r) .$$

Clearly, $\bigcap_{k=1}^{n} V(x_1,\rho_k,\delta)$ is an open set in $\left(X, \{\rho_\alpha\}\right)$. It follows that each open set in (X, S) is open in $\left(X, \{\rho_\alpha\}\right)$. The proof is complete.

Now let $\left(X, \{\rho_\alpha\}\right)$ be a uniform space and let S be the uniform structure generated by $\{\rho_\alpha\}$.

Let

$$L(\rho, r) = \{(x,y) \mid \rho(x,y) < r, r > 0, \rho \in S\} .$$

<u>Definition III-5</u>: The uniformity of the set X (relative to the collection $\{\rho_\alpha\}$) is the family $\mathcal{U}^*$ of subsets of $X \times X$ for which

$U \in \mathcal{U}^*$ if and only if there exists an $r > 0$

and

$\rho \in S$ such that $L(\rho, r) \subset U$.

The next theorem shows that our concept of a uniformity implies the axioms given by Kelly [6].

However, first some terminology is needed. Let

$$\Delta = \{\text{all pairs } (x,x) \mid x \varepsilon X\} .$$

Let U^{-1} denote the collection of all pairs (y,x) for which $(x,y) \varepsilon U$. If U_1 and U_2 are subsets of $X \times X$, we denote by $U_1 \circ U_2$ the family

$$\{(x,z) \mid \text{there exists } y \varepsilon X \text{ so that } (x,y) \varepsilon U_1,\ (y,z) \varepsilon U_2\}$$

Theorem III-5: If $(X, \{\rho_\alpha\})$ is a uniform space, then the uniformity of X (relative to $\{\rho_\alpha\}$) satisfies the following properties:

(1) If $U \varepsilon \mathcal{U}^*$, then $\Delta \subset U$.

(2) If $U \varepsilon \mathcal{U}^*$, then $U^{-1} \subset \mathcal{U}^*$.

(3) If $U_1 \varepsilon \mathcal{U}^*$, then there exists $U_2 \varepsilon \mathcal{U}^*$ such that $U_2 \circ U_2 \subset U_1$.

(4) If $U_1 \varepsilon \mathcal{U}^*$, $U_2 \varepsilon \mathcal{U}^*$, then $U_1 \cap U_2 \varepsilon \mathcal{U}^*$.

(5) If $U_1 \varepsilon \mathcal{U}^*$ and $U_1 \subset U_2 \subset X \times X$, then $U_2 \varepsilon \mathcal{U}^*$.

Proof: (1) If $U \varepsilon \mathcal{U}^*$, then there exists an $r > 0$ and $\rho \varepsilon S$ so that $L(\rho,r) \subset U$. Clearly, since $\rho(x,x) = 0 < r$, for every $x \varepsilon X$, we have $\Delta \subset L(\rho,r) \subset U$.

(2) Note that $(y,x) \varepsilon L(\rho,r)$ if and only if $(x,y) \varepsilon L(\rho,r)$. So $L(\rho,r) \subset U$ if and only if $L(\rho,r) \subset U^{-1}$.

(3) Let $L(\rho,r) \subset U_1$. Clearly $L(\rho, \frac{r}{2}) \varepsilon \mathcal{U}^*$. It follows easily that $L(\rho, \frac{r}{2}) \circ L(\rho, \frac{r}{2}) \subset L(\rho,r) \subset U_1$.

(4) Let $L(\rho_1, r_1) \subset U_1$ and $L(\rho_2, r_2) \subset U_2$. Now let $r = \min\{r_1, r_2\}$ and $\rho = \max\{\rho_1, \rho_2\}$. If

$$\rho(x,y) < r, \text{ then } \rho_k(x,y) \leq \rho(x,y) < r \leq r_k, \; k = 1,2,$$

and it follows that

$$L(\rho, r) \subset L(\rho_k, r_k) \subset U_k, \quad k = 1,2 .$$

Therefore $L(\rho, r) \subset U_1 \cap U_2$ and $U_1 \cap U_2 \in \mathcal{U}^*$.

(5) If $U_1 \subset U_2$ and $L(\rho, r) \subset U_1$, then $L(\rho, r) \subset U_2$ and $U_2 \in \mathcal{U}^*$.

The converse of Theorem III-5 is also true. A proof is given in Kelly [6].

The following sequence of theorems lead to the result that each proximity space is a uniform space. But first we make the following definition.

<u>Definition III-6</u>: Let

$$\left(X, \{\rho_\alpha\}\right) \text{ and } \left(X, \{\rho'_\beta\}\right)$$

be two uniform spaces with $f : X \to Y$. If f is one-to-one, onto, and both f and f^{-1} are uniformly continuous, then f is called a uniform isomorphism.

<u>Theorem III-6</u>: Each uniformly continuous function $f : \left(X, \{\rho_\alpha\}\right) \to \left(Y, \{\rho'_\beta\}\right)$ is continuous relative to the uniform topology.

<u>Proof</u>: Let $\bar{y} = f(\bar{x})$. Let $N(\bar{y}, \rho'_\beta, r)$ be a subbasis element of $\left(Y, \{\rho'_\beta\}\right)$ containing $\bar{y}$. By uniform continuity of f, there exists a finite subcollection $\{\rho_k\}$, $k = 1,2,\ldots, n$, of $\{\rho_\alpha\}$ and $\delta > 0$ such

that if $\rho = \max_{1 \leq k \leq n} \{\rho_k\}$, then $\rho'_\beta(f(x), \bar{y}) < r$ whenever $\rho(x,\bar{x}) < \delta$.

Let

$$O = \bigcap_{k=1}^{n} N(\bar{x}, \rho_k, \delta) .$$

Then O is open in $(X, \{\rho_\alpha\})$ and $\bar{y} \in f(O) \subset N(\bar{y}, \rho'_\beta, r)$. It follows that f is continuous.

<u>Theorem III-7</u>: Let $(X, \{\rho_\alpha\})$ be a uniform space; then $(X, \{\rho_\alpha\})$ is uniformly isomorphic to a subspace of the product of pseudometric spaces. The converse also holds.

<u>Proof</u>: Consider the collection of pseudometric spaces $(X, \rho_\alpha\}$. We wish to prove that the uniform space $(X, \{\rho_\alpha\})$ is uniformly isomorphic to the product space $Z = \prod \{(X, \rho_\alpha), \rho_\alpha \in \{\rho_\alpha\}\}$.

We consider the topology in each pseudometric space to be the uniform topology generated by the single pseudometric, and the topology in the product space to be the product topology.

Define $f : X \to Z$ as follows:

$f(x) = F_x$, where $F_x(\alpha) = x$ for each α in the indexing set A of $\{\rho_\alpha\}$. Note that the image of x under f is the point in Z all of whose coordinates are x. Now f is clearly 1-1 onto a subspace of Z.

We must show that f and f^{-1} are uniformly continuous. First we define for a finite collection $\rho_{\alpha_1}, \rho_{\alpha_2}, \ldots, \rho_{\alpha_n}$:

$$R\,[\alpha_1, \alpha_2, \ldots, \alpha_n;\ F_x, F_y] = \max_{1 \leq k \leq n} \left\{\rho_{\alpha_k}(F_x(\alpha_k), F_y(\alpha_k))\right\} .$$

It is easy to see that $R[\alpha_1, \alpha_2, \ldots, \alpha_n; F_x, F_y]$ is a pseudometric on Z. It is also easily verified that the collection of all pseudometrics R formed in this way generate the same topology on Z as the product topology. Hence the product topology is a uniform topology generated by the indicated collection of pseudometrics.

We now show uniform continuity of f: Let $R[\alpha_1, \alpha_2, \ldots, \alpha_n : F_x, F_y]$ and $\varepsilon > 0$ be given. Then, if we define

$$\rho = \max \{\rho_{\alpha_1}, \rho_{\alpha_2}, \ldots, \rho_{\alpha_n}\},$$

we have $\rho(x,y) < \varepsilon$ if and only if

$$\rho_{\alpha_i}(x,y) < \varepsilon \quad \text{for} \quad i = 1,2,\ldots, n.$$

Then

$$\rho_{\alpha_i}(F_x(\alpha_i), F_y(\alpha_i)) < \varepsilon, \; i=1,2,\ldots, n$$

and it follows that

$$R[\alpha_1, \alpha_2, \ldots, \alpha_n; F_x, F_y] < \varepsilon .$$

Thus f is uniformly continuous.

To show f^{-1} is continuous, we let ρ_α and $\varepsilon > 0$ be given. Then choose $R[\alpha; F_x, F_y]$, where

$$R[\alpha; F_x, F_y] = \rho_\alpha(F_x(\alpha), F_y(\alpha)) .$$

Then if $R[\alpha; F_x, F_y] < \varepsilon$, we have that $\rho_\alpha(F_x(\alpha), F_y(\alpha)) < \varepsilon$ and it follows that $\rho_\alpha(x,y) < \varepsilon$. Hence f^{-1} is uniformly continuous.

For the converse, we note that in the preceding proof, we have shown that the product of pseudometric spaces is a uniform space. Any subspace is then uniform and the result follows.

The following theorem is a crucial step in our development and will be used immediately to get our desired result relating proximity spaces and uniform spaces.

Theorem III-8: Let (X, T) be a topological space. Then (X, T) is completely regular if and only if (X, T) is homeomorphic to a subspace of the product of pseudometric spaces.

Proof: (1) Suppose first that there exists a homeomorphism h which maps (X, T) onto a subspace of a product of pseudometric spaces, $\prod_{\alpha \varepsilon A} (X_\alpha, \rho_\alpha)$. Let C be a closed set in (X, T) and $x \varepsilon X - C$. Since h is a homeomorphism, then $h(C)$ is closed and $h(x) \notin h(C)$. Thus there exists an open set O containing $h(x)$ for which $O \cap h(C) = \varphi$.

By definition of product topology, $O = \prod_{\alpha \varepsilon A} O_\alpha$, $O_\alpha \subset X_\alpha$, and for all but a finite number of the indexing set A, we have $O_\alpha = X_\alpha$. Since $O \cap h(C) = \varphi$, there exists at least one O_α for which $O_\alpha \cap (h(C))_\alpha = \varphi$. Now $(h(C))_\alpha$ is closed in (X_α, ρ_α). Consider the projection mapping

$$p_\alpha : \prod (X_\alpha, \rho_\alpha) \to (X_\alpha, \rho_\alpha)$$

defined by

$$p_\alpha(\{x_\alpha\}) = x_\alpha, \quad \{x_\alpha\} \varepsilon \prod (X_\alpha, \rho_\alpha).$$

Then p_α is continuous. By Theorem I-6, (X, ρ_α) is normal. If D

is a closed set in (X, ρ_α) and $y \varepsilon X - D$, then there exists an $r > 0$ such that

$$N(y,r) = \{z|\ \rho_\alpha(y,z) < r\} \subset X - D$$

It follows that $\overline{N(y, \frac{r}{2})} \subset X - D$. The normality of (X, ρ_α) applied now immediately gives complete regularity of (X, ρ_α).

Thus there exists a function $f : X_\alpha \to [0,1]$ such that $f[(h(x))_\alpha] = 0$, $f(y) = 1$ if $y \varepsilon (h(C))_\alpha$, and f is continuous from X_α onto $[0,1]$. It follows that the function

$$F = f\, p_\alpha\, h$$

is a continuous function mapping $X \to [0,1]$ (onto) for which $F(x) = 0$ and $F(y) = 1$ if $y \varepsilon C$. Hence (X, T) is completely regular.

(2) Suppose now that (X, T) is completely regular. We shall construct a collection of pseudometric spaces such that (X, T) is homeomorphic to their product.

For each closed set $C \subset (X, T)$ and each point $p \notin C$, by complete regularity, there exists a function

$$f_C^{(p)} \text{ for which } f_C^{(p)}(p) = 0,$$

$$f_C^{(p)}(x) = 1$$

if $x \varepsilon C$ and $f_C^{(p)}$ is continuous from (X, T) onto $[0, 1]$.

Define $\rho_C^{(p)}$ by

$$\rho_C^{(p)}(x,y) = |f_C^{(p)}(x) - f_C^{(p)}(y)|\ .$$

It is easily seen that $\rho_C^{(p)}$ is a pseudometric on $X \times X$. For each such pair (C, p), $p \notin C$, C closed, we form the pseudometric space $(X, \rho_C^{(p)})$. We consider the product space $\prod (X, \rho_C^{(p)})$ where the product is taken over all spaces formed in the indicated manner.

We assert that (X, T) is homeomorphic to a subspace of this product space. Define

$$h : (X, T) \to \prod (X, \rho_C^{(p)}) \text{ by:}$$

$h(x) = H_x$, where $H_x(p, C) = x$, i.e. the image of x under h is the point in $\prod (X, \rho_C^{(p)})$ all of whose coordinates are x. We will show that h is a homeomorphism. Clearly h is 1-1 and onto a subspace of $\prod (X, \rho_C^{(p)})$.

To show that h is continuous, let $H_x \varepsilon \prod (X, \rho_C^{(p)}) \cap h(X)$. Let O be an open set in the product topology of $\prod (X, \rho_C^{(p)})$ and $H_x \varepsilon O$. Then $O = \prod O_C^{(p)}$ where $O_C^{(p)} = X$ except in a finite number of cases. Since $O_C^{(p)}$ is open in $(X, \rho_C^{(p)})$ and since there are only finitely many of the sets $O_C^{(p)}$ to consider, there exists an $\varepsilon > 0$ such that

$$N(x, \rho_C^{(p)}, \varepsilon) \subset O_C^{(p)} \quad \text{for each } O_C^{(p)} .$$

Hence

$$\prod N(x, \rho_C^{(p)}, \varepsilon) \subset \prod O_C^{(p)}$$

Now since $f_C^{(p)}$ is continuous on (X, T), then for $\varepsilon > 0$ as determined above, there exists an open set $U_C^{(p)}$ (relative to T) in X containing x and such that $y \varepsilon U_C^{(p)}$ implies $|f_C^{(p)}(y) - f_C^{(p)}(x)| < \varepsilon$

Let $U = \bigcap U_C^{(p)}$, where the intersection is taken over the finite collection above. This U is open in (X, T).

Now if $y \varepsilon U$, then $y \varepsilon U_C^{(p)}$ and it follows that $y \varepsilon N(x, \rho_C^{(p)}, \varepsilon)$ for each pair (C, p). Hence $y \varepsilon O_C^{(p)}$ for each (C, p) and further $h(y) = H_y \varepsilon \prod O_C^{(p)}$. Therefore $h(U) \subset O$ and U is open. Thus h is continuous.

To show that h^{-1} is continuous, let $q \varepsilon O$, O open in (X,T). We must find an open set U in $\prod (X, \rho_C^{(p)})$ so that $h(q) \varepsilon U$ and $h^{-1}(U) \subset O$.

Consider $f_{X-O}^{(q)}$. Then

$$\rho_{X-O}^{(q)}(x, q) = f_{X-O}^{(q)}(x).$$

Let $U = \prod O_C^{(p)}$, where $O_C^{(p)} = X$ unless $p = q$ and $C = X - O$, in which case $O_{X-O}^{(q)} = N(q, \rho_{X-O}^{(q)}, 1)$. Then if

$$H_x = h(x) \varepsilon U, \text{ we have}$$

$$H_x(q, X\text{-}O) = x \varepsilon O_{X-O}^{(q)}.$$

Hence

$$\rho_{X-O}^{(q)}(q, x) < 1 \text{ so } f_{X-O}^{(q)}(x) < 1,$$

and it follows that $x \not\varepsilon X - O$ so $x \varepsilon O$. Thus h^{-1} is continuous. The proof is complete.

<u>Theorem III-9</u>: A topological space (X, T) is completely regular if and only if (X, T) is a uniform space.

<u>Proof</u>: This theorem is just a combination of Theorem III-7 and III-8.

Theorem III-10: Let (X, P) be a proximity space. Then (X, P) is a Hausdorff uniform space.

Proof: By Theorem II-8, every proximity space is a completely regular Hausdorff space and hence by Theorem III-9, a Hausdorff uniform space.

Theorem III-11: Let $\{(X_n, \rho_n)\}$ be a countable collection of pseudometric spaces; then the product space $\prod (X_n, \rho_n)$ is pseudometrizable.

Proof: By an elementary result in the theory of pseudometric spaces, we may as well assume that each ρ_n is bounded by 1. (See Kelly [6], Theorem 4-13).

Let $x = \{x_n\}$, $y = \{y_n\}$. Define:

$$\rho(x,y) = \sum_{n=1}^{\infty} \frac{1}{2^n} \rho_n(x_n, y_n),$$

which clearly converges. Then ρ is a pseudometric. The only property not completely trivial is the triangular inequality which is proved as follows:

$$\rho(x,z) = \sum_{n=1}^{\infty} \frac{1}{2^n} \rho_n(x_n, z_n) \leq \sum_{n=1}^{\infty} \frac{1}{2^n} [\rho_n(x_n,y_n) + \rho_n(y_n,z_n)]$$

and

$$\rho(x,z) \leq \sum_{n=1}^{\infty} \frac{1}{2^n} \rho_n(x_n,y_n) + \sum_{n=1}^{\infty} \frac{1}{2^n} \rho_n(y_n,z_n) = \rho(x,y) + \rho(y,z)$$

We must show that ρ generates the same topology on $\prod (X_n,\rho_n)$ as the product topology. Let O be an open set in X (relative to ρ)

and let $x \varepsilon O$. Then there exists a neighborhood $V(x, \rho, \frac{1}{2^p})$ for some integer p such that $V(x, \rho, \frac{1}{2^p}) \subset O$.

Now the set

$$U = \{y | \rho_n(x_n, y_n) < \frac{1}{2^{p+1}}\}, \ n=1,\ldots,p+1$$

is an open set in $\prod (X_n, \rho_n)$. Let $y \varepsilon U$. Then

$$\rho(x,y) = \sum_{n=1}^{\infty} \frac{1}{2^n} \rho_n(x_n, y_n) \leq \sum_{n=1}^{p+1} \frac{1}{2^n} \frac{1}{2^{p+1}} + \sum_{n=p+2}^{\infty} \frac{1}{2^n}$$

and so

$$\rho(x,y) \leq \frac{1}{2^{p+1}} + \sum_{K=1}^{\infty} \frac{1}{2^K} \frac{1}{2^{p+1}} \leq \frac{1}{2^{p+1}} + \frac{1}{2^{p+1}} = \frac{1}{2^p} .$$

So $y \varepsilon V(x, \rho, \frac{1}{2^p})$ and thus $x \varepsilon U \subset V(x, \rho, \frac{1}{2^p}) \subset O$. Therefore, since x was arbitrary in O, O is open in the product topology.

Conversely, let O be a subbasis element in the product topology. Then $O = \prod O_n$, where for all but a finite number of the sets O_n, we have $O_n = X_n$. Let $x \varepsilon O$. Then clearly there exists an $r > 0$ such that for each O_n,

$$N(x_n, \rho_n, r) \subset O_n .$$

Now

$$\rho(x,y) \geq \frac{1}{2^n} \rho_n(x_n, y_n) .$$

Hence $N(x, \rho, \frac{r}{2^n}) \subset O$. Therefore O is open in (X, ρ). Since each subbasis element of the product topology is open in (X, ρ), it follows that each set open in the product topology is open in (X, ρ).

<u>Theorem III-12</u>: Let $(X,\{\rho_n\})$ be a uniform space generated by a countable collection of pseudometrics $\{\rho_n\}$. Then $(X, \{\rho\})$ is pseudometrizable.

<u>Proof</u>: The proof of Theorem III-7 shows that $(X, \{\rho_n\})$ is uniformly isomorphic to a subspace of the product of a countable collection of pseudometric spaces. By Theorem III-11, this product is pseudometrizable. It then follows that $(X, \{\rho_n\})$ is pseudometrizable.

<u>Corollary</u>: Let $(X, \{\rho_n\})$ be as in Theorem III-12 with the added hypothesis that $(X, \{\rho_n\})$ is Hausdorff. Then $(X, \{\rho_n\})$ is metrizable.

<u>Proof</u>: Let ρ be a pseudometric which generates the topology of $(X, \{\rho_n\})$. The existence of ρ is guaranteed by Theorem III-12. Since $(X, \{\rho_n\})$ is Hausdorff, for any two points x and y (distinct) of X, there exist $N(x,\rho,\varepsilon_1)$ and $N(y,\rho,\varepsilon_2)$ for which

$$N(x,\rho,\varepsilon_1) \cap N(y,\rho,\varepsilon_2) = \phi.$$

Thus

$$x \notin N(y,\rho,\varepsilon_2) .$$

It follows that

$$\rho(x,y) \geq \varepsilon_2 > 0 .$$

So if $x \neq y$, $\rho(x,y) > 0$. Therefore, if $\rho(x,y) = 0$, then $x = y$. Hence ρ is actually a metric.

We have seen (Theorem III-10) that each proximity space is a

Hausdorff uniform space. Our next theorem shows that the converse is also true.

Definition III-7: A collection $\{\rho_\alpha\}$, $\alpha \varepsilon I$ (an indexing set), of pseudometrics on X is called total if and only if for each pair of distinct points $x, y \varepsilon X$, there exists a pseudometric $\rho \varepsilon (\rho_\alpha)$ for which $\rho(x,y) > 0$.

Note: If $\left(X, \{\rho_\alpha\}\right)$ is a Hausdorff uniform space, then it follows easily that $\{\rho_\alpha\}$ is total.

For each $\alpha \varepsilon I$, let

$$D_\alpha(A,B) = \inf \{\rho_\alpha(x,y) | x \varepsilon A, \ y \varepsilon B \},$$

where A and B are any two nonempty subsets of X. We define

$$(A,B) \varepsilon P \text{ if and only if } D_\alpha(A,B) = 0 \text{ for all } \alpha \varepsilon I.$$

Theorem III-13: Let $\left(X,\{\rho_\alpha\}\right)$ be a Hausdorff uniform space. Using P as defined above, (X, P) is a proximity space and the topologies in (X, P) and $\left(X, \{\rho_\alpha\}\right)$ are identical.

Proof: That P is indeed a proximity relation follows the same pattern as Theorem II-9. We will indicate part of the proof.

Suppose $(A, B) \not\varepsilon P$. Then there exists $\alpha \varepsilon I$ for which $D_\alpha(A, B) = h > 0$. Then

$$A \subset \{x | D_\alpha(x,A) < \tfrac{h}{2} \} = E,$$

and $B \subset X - E$. It follows that $(A, X - E) \not\varepsilon P$ and $(B, E) \not\varepsilon P$.

Since $\{\rho_\alpha\}$, $\alpha \varepsilon I$, is total, it follows that $x = y$ whenever $(x,y) \varepsilon P$.

It is clear that a point $x \varepsilon X$ is a limit point of a set $A \subset X$ relative to the uniform topology if and only if x is a limit point of A relative to the topology induced by the proximity relation P. Hence the two topologies are equivalent.

It follows from this last theorem and Theorem III-10 that the concepts of a Hausdorff uniform space and a proximity space are identical. The proximity spaces are not only a proper subclass of the uniform spaces, but they are precisely the <u>Hausdorff</u> uniform spaces.

CHAPTER IV

GENERALIZED METRIC SPACES

Our third generalization of a metric space is accomplished by retaining the concept of a distance function; however, we generalize the range of the distance function and obtain rather surprising results.

We note here that generalizations of this sort have been made with varying degrees of generality. The papers of David Ellis [1] and G. B. Price [8] are of interest here. In these papers, the main restriction on the degree of generalization is the necessity to have some sort of order relation in the range space of the distance function. Thus, the writers generally require that the range of the distance function be a subset of some partially ordered space.

Definition IV-1: A set A is partially ordered by a relation $\leq$ provided that if $x, y, z \in A$, then

(1) $x \leq y$, $y \leq x$ imply $x = y$, and (2) if $x \leq y$, $y \leq z$, then $x \leq z$.

Definition IV-2: A set I is a directed set provided that I is partially ordered and satisfies the further property: Given i, j in I, there exists a k in I for which $i \leq k$, $j \leq k$.

We shall make use of the directed set I as an indexing set. Let (R,d) denote the space of real numbers with the metric topology generated by

$$d(x,y) = |x - y|, \qquad x,y \in R.$$

We shall require that our distance function have values in a product space constructed from R.

For I as described above, we consider the space $G = \prod_{i \in I} (R,d)$.

We denote an element of G by (r_i). Define an order relation in G as follows:

(*) $(r_i) \leq (r_i')$ provided if $r_j > r_j'$ for some j, then there exists a k, $k < j$, such that $r_k < r_k'$ and $r_m \leq r_m'$ for each $m < k$.

If $(r_i) \leq (r_i')$ and $(r_i) \neq (r_i')$, then we say that $(r_i) < (r_i')$.

<u>Theorem IV-1</u>: G is partially ordered by the relation given in (*).

<u>Proof</u>: (1) Suppose $(r_i) \leq (r_i')$ and $(r_i') \leq (r_i)$. Then if $r_j > r_j'$ for some j, there exists k, $k < j$, such that $r_k < r_k'$ and $r_m \leq r_m'$ for all $m < k$. Using $(r_i') \leq (r_i)$, we then have, since $r_k' > r_k$, that there exists a p, $p < k$, such that $r_p' < r_p$. But this is a contradiction since $r_m \leq r_m'$ for each $m < k$. Thus the assumption $r_j > r_j'$ for some j is false. Hence $r_k \leq r_k'$ for each k.

By similar argument, we find that $r_k' \leq r_k$ for each k. Hence $r_k = r_k'$ for each k and thus $(r_k) = (r_k')$.

(2) Suppose $(r_i \leq r_i')$ and $(r_i') \leq (r_i'')$. We wish to show that $(r_i) \leq (r_i'')$. If $(r_i) = (r_i')$ and $(r_i') = (r_i'')$, the proof is trivial.

Suppose then that $r_j > r_j''$ for some j. Then either $r_j > r_j'$ or $r_j' > r_j''$ or both. Suppose, for definiteness, that $r_j > r_j'$. Then there exists k_1, $k_1 < j$, such that $r_{k_1} < r_{k_1}'$ and $r_m \leq r_m'$ for all $m < k_1$.

Now we consider two cases.

Case I: Suppose $r_m' \leq r_m''$ for all $m \leq k_1$. Then $r_{k_1} < r_{k_1}''$ and $r_m \leq r_m''$ for all $m < k_1$. Thus $(r_i) \leq (r_i'')$.

Case II: Suppose $r_{k_2}' > r_{k_2}''$ for some $k_2 \leq k_1$. Then since $(r_i') \leq (r_i'')$, we have that there exists a k, $k < k_2$ such that $r_k' < r_k''$ and $r_m' \leq r_m''$ for all $m < k$. Therefore $r_k < r_k''$ and $r_m \leq r_m''$ for each $m \leq k$. Hence $(r_i) \leq (r_i'')$.

So our range space G is partially ordered.

We define addition in G by $(r_i) + (r_i') = (r_i + r_i')$.

Definition IV-3: A generalized pseudometric $g : X \times X \to G$ is a function with the properties

(1) $g(x,x) = (0)$,

(2) $g(x,y) \geq (0)$,

(3) $g(x,y) = g(y,x)$,

(4) $g(x,z) \leq g(x,y) + g(y,z)$,

for each x,y,z in X.

We note that these properties correspond to those of a pseudometric. We could have added the condition if $g(x,y) = (0)$, then $x = y$, to give the metric properties, but we will be more interested in the generalized pseudometric as defined above. It will be clear that this additional metric property forces the property of Hausdorff in the topological space we obtain. It will also be clear that this additional metric property would give equivalence of generalized metric spaces and Hausdorff uniform spaces.

We proceed to define a topology in X using the generalized distance function. First, we topologize the range space G.

Let $\{i_1, i_2, \ldots, i_n\}$ be any finite subset of I; a neighborhood of a point (r_i) in G is defined as follows:

$$N_\varepsilon[(r_i)] = \{(r'_i) \,\big|\, |r_{i_k} - r'_{i_k}| < \varepsilon, \quad k = 1,2, \ldots, n, \ \varepsilon > 0\} .$$

It is easily seen that the collection of all neighborhoods of this form is a basis for a topology in X. We use N_ε in what follows to mean $N_\varepsilon((0))$.

Of course, N_ε depends on which finite subset of I is chosen, but we have this understood and for notational convenience suppress any indication of that dependence.

We are now in a position to define a topology in the space X. Given a neighborbood, N_ε of (0) in G, we define a neighborhood $U_{N_\varepsilon}(x)$ for $x \varepsilon X$ as $U_{N_\varepsilon}(x) = \{y | g(x,y) \ \varepsilon \ N_\varepsilon\}$.

<u>Theorem IV-2</u>: The neighborhoods $U_{N_\varepsilon}(x)$ defined above qualify as a basis for a topology in X.

P r o o f: It is clear that the union of all such neighborhoods is equal to the set X.

Now let $U_{N_{\varepsilon_1}}(x_1)$ and $U_{N_{\varepsilon_2}}(x_2)$ be two such neighborhoods whose intersection is nonvoid and assume y is an element of the intersection. Then clearly,

$$g(x_1,y) \ \varepsilon \ N_{\varepsilon_1} \quad \text{and} \quad g(x_2,y) \ \varepsilon \ N_{\varepsilon_2} .$$

Now N_{ε_1} and N_{ε_2} are sets of the form

$$N_{\varepsilon_1} = \{(r_i) \big| |r_{i_k}| < \varepsilon_1, \quad k = 1,2, \ldots, n\}$$

$$N_{\varepsilon_2} = \{(r_i') \big| |r_{i_k}'| < \varepsilon_2, \quad k = 1,2, \ldots, m\}.$$

Note that the finite subsets of I used in N_{ε_1} and N_{ε_2} have no relation to each other.

Let

$$(a_i) = g(x_1,y) \ \varepsilon \ N_{\varepsilon_1} ; \quad \text{and} \quad (b_i) = g(x_2,y) \ \varepsilon \ N_{\varepsilon_2} .$$

Then

$$|a_{i_k}| < \varepsilon_1, \quad k = 1,2, \ldots, n; \quad \text{and} \quad |b_{i_j}| < \varepsilon_2, \quad j = 1,2, \ldots, m .$$

Let

$$\varepsilon_3 = \min_{1 \leq k \leq n} \{\varepsilon_1 - |a_{i_k}|\} > 0$$

and

$$\varepsilon_4 = \min_{1 \leq j \leq m} \{\varepsilon_2 - |b_{i_j}|\} > 0 .$$

Now let $\varepsilon = \min\{\varepsilon_3, \varepsilon_4\}$. Then consider N_ε defined as

$$N_\varepsilon = \{(r_i) \big| |r_{i_p}| < \varepsilon, \quad p = 1,2, \ldots, M\},$$

where p is allowed to run over the finite subset of I consisting of the union of the two finite subsets of I used in N_{ε_1} and N_{ε_2}. Thus

$$\max\{n,m\} \leq M \leq n + m.$$

It follows easily now that $N_\varepsilon \subset N_{\varepsilon_1} \cap N_{\varepsilon_2}$. We further claim that $U_{N_\varepsilon}(y) \subset U_{N_{\varepsilon_1}}(x_1) \cap U_{N_{\varepsilon_2}}(x_2)$. For is $z \varepsilon U_{N_\varepsilon}(y)$, then $g(y,z) \varepsilon N_\varepsilon \subset N_{\varepsilon_1} \cap N_{\varepsilon_2}$. Therefore $Z \varepsilon U_{N_{\varepsilon_1}}$ and $z \varepsilon U_{N_{\varepsilon_2}}$. Then

$$U_{N_\varepsilon}(y) \subset U_{N_{\varepsilon_1}}(x_1) \cap U_{N_{\varepsilon_2}}(x_2)$$

and the neighborhoods $U_{N_\varepsilon}(x)$ define a topology in X.

<u>Definition IV-4</u>: The topological space formed by imposing the topology generated by the generalized pseudometric g on the set X is called a <u>generalized pseudometric space</u> and is denoted by (X,g).

We are now in a position to prove the following important theorem.

<u>Theorem IV-3</u>: The generalized pseudometric space (X,g) is a uniform space.

<u>Proof</u>: We will show that we can define a uniformity in X using the pseudometric g. Let

$$U_N = \{(x,y) \mid g(x,y) \in N_\varepsilon\}.$$

Define the uniformity $\mathcal{U}$ in X as follows:

$U \in \mathcal{U}$ if and only if there exists N_ε such that $U_{N_\varepsilon} \subset U$.

We will show that the uniformity thus defined satifies the properties of Theorem III-5.

(a) Clearly $\Delta \subset U_{N_\varepsilon}$ for every N_ε since $g(x,x) = (0)$.

(b) Suppose $U \in \mathcal{U}$. Then there exists U_{N_ε} such that $U_{N_\varepsilon} \subset U$. Now $U_{N_\varepsilon} = U_{N_\varepsilon}^{-1}$. Further, $U_{N_\varepsilon}^{-1} \subset U^{-1}$ and so $U_{N_\varepsilon} \subset U^{-1}$. Hence $U^{-1} \in \mathcal{U}$.

(c) Let $U \in \mathcal{U}$. Then there exists U_{N_ε} such that $U_{N_\varepsilon} \subset U$. Then $U_{N_{\varepsilon/2}} \circ U_{N_{\varepsilon/2}} \subset U_{N_\varepsilon} \subset U$, where the same finite subset of I is used in defining $U_{N_{\varepsilon/2}}$ and U_{N_ε}. For if $(x,y) \in U_{N_{\varepsilon/2}}$ and $(y,z) \in U_{N_{\varepsilon/2}}$, we have $g(x,y) \in N_{\varepsilon/2}$ and $g(y,z) \in N_{\varepsilon/2}$. Thus $g(x,y) + g(y,z) \in N_\varepsilon$ and $g(x,z) \in N_\varepsilon$.

(d) Let $U \in \mathcal{U}$, $V \in \mathcal{U}$. Then there exists $U_{N_{\varepsilon_1}} \subset U$ and $U_{N_{\varepsilon_2}} \subset V$. Let $\varepsilon = \min\{\varepsilon_1, \varepsilon_2\}$. Then $U_{N_\varepsilon} \subset U \cap V$. Thus $U \cap V \in \mathcal{U}$.

(e) Let $U \in \mathcal{U}$ and $U \subset V \subset X \times X$. Then there exists U_{N_ε} for which $U_{N_\varepsilon} \subset U \subset V$ and thus $V \in \mathcal{U}$.

It is not hard to show by standard techniques that the topology generated in X by this uniformity is equivalent to the original topology in (X,g). See Kelly [6], Chapter 6.

An independent proof of this last theorem can be given easily by first showing that (X,g) is completely regular, from which it follows that (X,g) is a uniform space. This method of proof mimics Theorem II-8.

Theorem IV-4: If $(X,\{\rho_\alpha\})$ is a uniform space, then $(X,\{\rho_\alpha\})$ is homeomorphic to a generalized pseudometric space.

Proof: By Theorem III-7, $(X,\{\rho_\alpha\})$ is uniformly isomorphic to a subspace of the product of the pseudometric spaces (X,ρ_α). Denote the indexing set over which α ranges by M_o. Let $\mathcal{M}_o$ be the set of all finite subsets of M_o.

We define the following generalized pseudometric:

$$g(x,y) = (r_M), \quad \text{where } M \in \mathcal{M}_o$$

and

$$r_M = \max \{\rho_\alpha(x,y) \mid \alpha \in M\}.$$

We assert that (X,g) is a generalized pseudometric space. We first partially order $\mathcal{M}_o$ by defining for $M_1, M_2 \in \mathcal{M}_o$

$$M_1 \leq M_2 \text{ if and only if } M_1 \subset M_2.$$

It follows easily that this definition satisfies the properties of Definition IV-2.

It is also clear that the range of g is a subset of G as defined earlier.

We must show that g is a generalized pseudometric.

(1) $\rho_\alpha(x,x) = 0$ for each $\alpha \varepsilon M$ implies that

$$r_M = \max\{\rho_\alpha(x,x) | \alpha \varepsilon M\} = 0.$$

Hence

$$g(x,x) = (0).$$

(2) $\rho_\alpha(x,y) \geq 0$ implies that $r_M \geq 0$ and hence

$$g(x,y) \geq (0).$$

(3) Since $\rho_\alpha(x,y) = \rho_\alpha(y,x)$, we have

$$\max\{\rho_\alpha(x,y) | \alpha \varepsilon M\} = \max\{\rho_\alpha(y,x) | \alpha \varepsilon M\}.$$

Thus

$$g(x,y) = g(y,x) .$$

(4) Since $\rho_\alpha(x,y) + \rho_\alpha(y,z) \geq \rho_\alpha(x,z)$, then

$$\max\{\rho_\alpha(x,z) | \alpha \varepsilon M\} \leq \max\{\rho_\alpha(x,y) | \alpha \varepsilon M\} + \max\{\rho_\alpha(y,z) | \alpha \varepsilon M\},$$

so

$$g(x,z) \leq g(x,y) + g(y,z)$$

and g is a generalized pseudometric.

We note that a neighborhood

$$U_{N_\varepsilon}(x) = \{y | g(x,y) \varepsilon N_\varepsilon\} \text{ in } (X,g)$$

is the collection of all $y \varepsilon X$ for which $\rho_\alpha(x,y) < \varepsilon$ for a finite subset of $\{\rho_\alpha\}$. Hence the neighborhood $U_{N_\varepsilon}(x)$ in (X,g) corresponds precisely to a neighborhood of $\{x_\alpha | x_\alpha = \dot{x}\} \varepsilon \prod_{\alpha \varepsilon M_o} (X,\rho_\alpha)$ in the product topology. It follows then that (X,g) is homeomorphic to the subspace of $\prod_{\alpha \varepsilon M} (X,\rho_\alpha)$ which consists of only those points all of whose coordinates are equal. But $(X,\{\rho_\alpha\})$ is uniformly isomorphic to this same subspace. Hence $(X,\{\rho_\alpha\})$ and (X,g) are homeomorphic.

These last two theorems imply the equivalence of uniform spaces and generalized pseudometric spaces in the sense that we have defined them.

An easy corollary to the preceding work is that (X,g) is a generalized <u>metric</u> space if and only if (X,g) is homeomorphic to a <u>Hausdorff</u> uniform space. But Hausdorff uniform spaces are equivalent to proximity spaces. Thus

<u>Theorem IV-5</u>: Let (X,P) be a proximity space. Then (X,P) is homeomorphic to a generalized <u>metric</u> space. The converse also holds.

Most of the results of this chapter appear in the paper by Kelisch [5].

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